Proving Non-regularity

Lecture 6

Thursday, January 31, 2019

LaTeXed: January 31, 2019 15:54

Regular Languages, DFAs, NFAs

Theorem

Languages accepted by DFAs, NFAs, and regular expressions are the same.

$$\begin{tikzcd} \text{Reg Exp} \arrow[r] & \text{NFA} \arrow[r] & \text{DFA} \arrow[ll, bend left] \end{tikzcd}$$

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- Number of languages is *uncountably infinite*

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- Each DFA M can be represented as a string over a finite alphabet Σ by appropriate encoding
- Hence number of regular languages is *countably infinite*
- Number of languages is *uncountably infinite*
- Hence there must be a non-regular language!

How to prove non-regularity?

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Lemma

Consider three strings $x, y, w \in \Sigma^*$.
$M = (Q, \Sigma, \delta, s, A)$: DFA for language $L \subseteq \Sigma^*$.
If $\delta^*(s, xw) \in A$ and $\delta^*(s, yw) \notin A$ then $\delta^*(s, x) \neq \delta^*(s, y)$.

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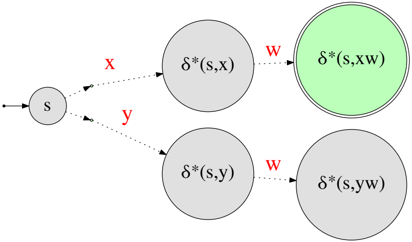
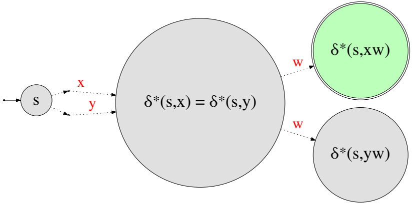
$\implies A \ni \delta^*(s, xw) \notin A$. Impossible! $\square$

Proof by figures

Possible	Not possible
	

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Intuition: Any program to recognize L seems to require counting number of zeros in input which cannot be done with fixed memory.

How do we formalize intuition and come up with a formal proof?

Proof by Contradiction

- Suppose L is regular. Then there is a DFA M such that $L(M) = L$.
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What states does M reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \leq i < j \leq n$.
That is, M is in the same state after reading 0^i and 0^j where $i \neq j$.

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M should accept $0^i 1^i$ but then it will also accept $0^j 1^i$ where $i \neq j$.

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M should accept $0^i 1^i$ but then it will also accept $0^j 1^i$ where $i \neq j$. This contradicts the fact that M accepts L. Thus, there is no DFA for L.

Generalizing the argument

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Example: 000 and 0000 are indistinguishable with respect to the language $L = \{w \mid w \text{ has } 00 \text{ as a substring}\}$

Wee Lemma

Lemma

Suppose $L = L(M)$ *for some* DFA $M = (Q, \Sigma, \delta, s, A)$ *and suppose* x, y *are distinguishable with respect to* L. *Then* $\delta^*(s, x) \neq \delta^*(s, y)$.

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Proof.

Since x, y are distinguishable let w be the distinguishing suffix. If $\delta^*(s, x) = \delta^*(s, y)$ then M will either accept both the strings xw, yw, or reject both. But exactly one of them is in L, a contradiction. □

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Suppose there is a DFA $M = (Q, \Sigma, \delta, s, A)$ that accepts L. Let $|Q| = n$.

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Proof.

Suppose there is a DFA $M = (Q, \Sigma, \delta, s, A)$ that accepts L. Let $|Q| = n$.
If $n < |F|$ then by pigeon hole principle there are two strings $x, y \in F$, $x \neq y$ such that $\delta^*(s, x) = \delta^*(s, y)$ but x, y are distinguishable.

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Implies that there is w such that exactly one of xw, yw is in L.
However, M's behavior on xw and yw is exactly the same and hence M will accept both xw, yw or reject both. A contradiction. □

Infinite Fooling Sets

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Corollary

If L has an infinite fooling set F then L is not regular.

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Proof.

Suppose for contradiction that $L = L(M)$ for some DFA M with n states.
Any subset F' of F is a fooling set. (Why?) Pick $F' \subseteq F$ arbitrarily such that $|F'| > n$. By preceding theorem, we obtain a contradiction. $\square$

Examples

- $\{0^k1^k \mid k \geq 0\}$

$F = \{0^i \mid i \geq 0\}$

$0^i1^i \quad 0^j1^i$

Examples

- $\{0^k 1^k \mid k \geq 0\}$
- {bitstrings with equal number of 0s and 1s}

$F = \{0^i \mid i \geq 0\}$

$0^i 1^i \in L$

$0^j 1^i \notin L$

$j \neq i$

Examples

- $\{0^k 1^k \mid k \geq 0\}$ $= L_1$
- {bitstrings with equal number of 0s and 1s}
- $\{0^k 1^\ell \mid k \neq \ell\}$ $= L_2$

$L_2 = \overline{L_1} \cap 0^* 1^*$

$F = \{0^i \mid i \geq 0\}$

$0^i 1^i \notin L \quad 0^j 1^i \in L$

Examples

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- {bitstrings with equal number of 0s and 1s}
- $\{0^k 1^\ell \mid k \neq \ell\}$
- $\{0^{k^2} \mid k \geq 0\}$ $= \{\varepsilon, 0, 0^4, 0^9, 0^{16} \dots\}$

$F = \{0^i \mid i \geq 3\}$

$k = i-1$

$i > j$

$\rightarrow 0^i \qquad 0^j$

$k^2 < i^2 - i + j < (k+1)^2$

0^{i^2-i}

$(i-1)^2 < i^2 - i + j < i^2$

$0^{i^2} \qquad 0^{i^2-i+j} \neq k^2 < 1$

$i^2 - 2i + 1 < i^2 - i + j$

$\notin L$

$-i+1 < j \Rightarrow i > 1-j$

Examples

- $\{ww^R \mid w \in \Sigma^*\}$

$F = \{1^i00 \mid i \geq 1\}$

$1^i001^i \in L$

$1^j001^i \notin L$

$F = \{1^i \mid i \geq 1\}$

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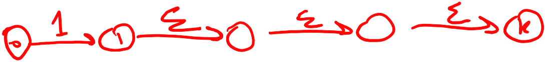
- $\{www \mid w \in \Sigma^*\}$

Exponential gap between NFA and DFA size

$L_k = \{w \in \{0,1\}^* \mid w$ has a 1 k positions from the end$\}$

$(0+1)^* 1 (0+1)^k$



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Claim

$F = \{w \in \{0,1\}^* : |w| = k\}$ *is a fooling set of size* 2^k *for* L_k.

Why? $k=5$

00000
00001
00010
⋮

0 0 0 1 0 0
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kth

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Why?

- Suppose $a_1 a_2 \ldots a_k$ and $b_1 b_2 \ldots b_k$ are two distinct bitstrings of length k
- Let i be first index where $a_i \neq b_i$
- $y = 0^{k-i-1}$ is a distinguishing suffix for the two strings

How do pick a fooling set

How do we pick a fooling set F?

- If x, y are in F and $x \neq y$ they should be distinguishable! Of course.
- All strings in F except maybe one should be prefixes of strings in the language L.
  For example if $L = \{0^k 1^k \mid k \geq 0\}$ do not pick 1 and 10 (say). Why?

Part I

Non-regularity via closure properties

Non-regularity via closure properties

$L = \{\text{bitstrings with equal number of 0s and 1s}\}$

$L' = \{0^k 1^k \mid k \geq 0\}$

Suppose we have already shown that L' is non-regular. Can we show that L is non-regular without using the fooling set argument from scratch?

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$L' = L \cap L(0^*1^*)$

Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

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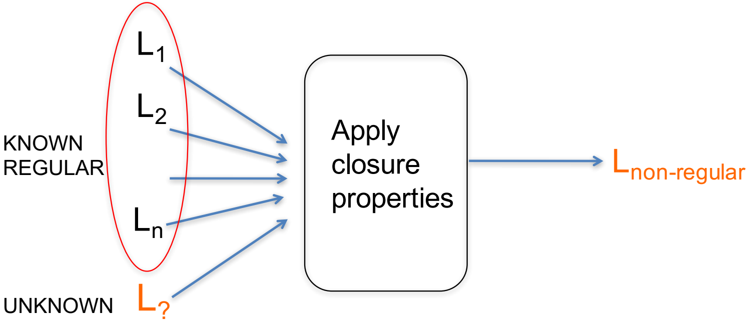
Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

Suppose L is regular. Then since $L(0^*1^*)$ is regular, and regular languages are closed under intersection, L' also would be regular. But we know L' is not regular, a contradiction.

Non-regularity via closure properties

General recipe:



Proving non-regularity: Summary

- Method of distinguishing suffixes. To prove that L is non-regular find an infinite fooling set.
- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.
- Pumping lemma. We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.

Part II

Myhill-Nerode Theorem

Indistinguishability

Recall:

Definition

For a language L over Σ and two strings $x, y \in \Sigma^*$ we say that x and y are distinguishable with respect to L if there is a string $w \in \Sigma^*$ such that exactly one of xw, yw is in L. x, y are indistinguishable with respect to L if there is no such w.

Given language L over Σ define a relation $\equiv_L$ over strings in Σ^* as follows: $x \equiv_L y$ iff x and y are indistinguishable with respect to L.

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Therefore, $\equiv_L$ partitions Σ^* into a collection of equivalence classes $X_1, X_2, \ldots,$

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Claim

Let x, y be two distinct strings. If x, y belong to the same equivalence class of $\equiv_L$ then x, y are indistinguishable. Otherwise they are distinguishable.

Corollary

If $\equiv_L$ is finite with n equivalence classes then there is a fooling set F of size n for L. If $\equiv_L$ is infinite then there is an infinite fooling set for L.

Myhill-Nerode Theorem

Theorem (Myhill-Nerode)

L is regular $\iff$ $\equiv_L$ has a finite number of equivalence classes. If $\equiv_L$ is finite with n equivalence classes then there is a DFA M accepting L with exactly n states and this is the minimum possible.

Corollary

A language L is non-regular if and only if there is an infinite fooling set F for L.

Algorithmic implication: For every DFA M one can find in polynomial time a DFA M' such that $L(M) = L(M')$ and M' has the fewest possible states among all such DFAs.