Kartsuba's Algorithm and Linear Time Selection

Lecture 11

Thursday, February 21, 2019

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Part I

Fast Multiplication

Multiplying Numbers

Problem Given two n-digit numbers x and y, compute their product.

Grade School Multiplication

Compute “partial product” by multiplying each digit of y with x and adding the partial products.

$$\begin{array}{r} 3141 \\ \times 2718 \\ \hline 25128 \\ 3141 \\ 21987 \\ 6282 \\ \hline 8537238 \end{array}$$

Time Analysis of Grade School Multiplication

1. Each partial product: $\Theta(n)$
2. Number of partial products: $\Theta(n)$
3. Addition of partial products: $\Theta(n^2)$
4. Total time: $\Theta(n^2)$

Divide and Conquer

Assume n is a power of 2 for simplicity and numbers are in decimal.

Split each number into two numbers with equal number of digits

1. $x = x_{n-1}x_{n-2}\ldots x_0$ and $y = y_{n-1}y_{n-2}\ldots y_0$

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2. $x = x_{n-1}\dots x_{n/2}0\dots 0 + x_{n/2-1}\dots x_0$

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2. $x = x_{n-1}\dots x_{n/2}0\dots 0 + x_{n/2-1}\dots x_0$
3. $x = 10^{n/2}x_L + x_R$ where $x_L = x_{n-1}\dots x_{n/2}$ and $x_R = x_{n/2-1}\dots x_0$

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2. $x = x_{n-1}\ldots x_{n/2}0\ldots 0 + x_{n/2-1}\ldots x_0$
3. $x = 10^{n/2}x_L + x_R$ where $x_L = x_{n-1}\ldots x_{n/2}$ and $x_R = x_{n/2-1}\ldots x_0$
4. Similarly $y = 10^{n/2}y_L + y_R$ where $y_L = y_{n-1}\ldots y_{n/2}$ and $y_R = y_{n/2-1}\ldots y_0$

Example

$$
\begin{aligned}
\overset{x}{\underline{1234}} \times \overset{\overset{y}{\downarrow}}{\underline{5678}} &= \overset{10^2 x_L + x_R}{(100 \times \underline{12} + 34)} \times \overset{10^2 y_L + y_R}{(100 \times \underline{56} + 78)} \\
&= 10000 \times 12 \times 56 \\
&\quad +100 \times (\underline{12} \times \underline{78} + \underline{34} \times \underline{56}) \\
&\quad +\underline{34} \times \underline{78}
\end{aligned}
$$

Divide and Conquer

Assume n is a power of 2 for simplicity and numbers are in decimal.

1. $x = x_{n-1}x_{n-2}\dots x_0$ and $y = y_{n-1}y_{n-2}\dots y_0$
2. $x = 10^{n/2}x_L + x_R$ where $x_L = x_{n-1}\dots x_{n/2}$ and $x_R = x_{n/2-1}\dots x_0$
3. $y = 10^{n/2}y_L + y_R$ where $y_L = y_{n-1}\dots y_{n/2}$ and $y_R = y_{n/2-1}\dots y_0$

Therefore

$$\begin{aligned} xy &= (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \\ &= 10^n \underline{x_L y_L} + 10^{n/2}(\underline{x_L y_R} + \underline{x_R y_L}) + \underline{x_R y_R} \end{aligned}$$

Time Analysis

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4 recursive multiplications of number of size $n/2$ each plus 4 additions and left shifts (adding enough 0's to the right)

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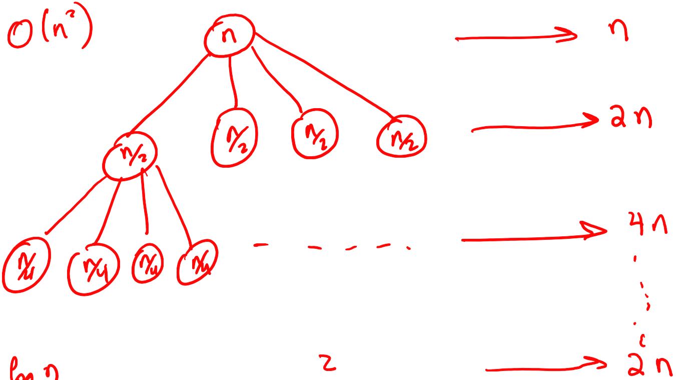
4 recursive multiplications of number of size $n/2$ each plus 4 additions and left shifts (adding enough 0's to the right)

$$T(n) = 4T(n/2) + O(n) \qquad T(1) = O(1)$$

Recursion tree analysis

$$T(n) = 4T(n/2) + O(n) \qquad T(1) = O(1)$$

$$\sum_{i=0}^{\log_2 n - 1} 2^i n = O(n^2)$$



$$2^{\log_2 n} \times n = n \times n = n^2$$

Recursion tree analysis

$$T(n) = 4T(n/2) + O(n) \qquad T(1) = O(1)$$

$T(n) = \Theta(n^2)$. No better than grade school multiplication!

A Trick of Gauss

Carl Friedrich Gauss: 1777–1855 “Prince of Mathematicians”

Observation: Multiply two complex numbers: $(a + \overset{\overset{10^{n/2}}{\downarrow}}{b}i)$ and $(c + di)$

$$(a + bi)(c + di) = \underline{ac} - \underline{bd} + (\underline{ad} + \underline{bc})i$$

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How many multiplications do we need?

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$$(a+b)(c+d) - ac - bd$$

$$\cancel{ac} + ad + bc + \cancel{bd} - \cancel{ac} - \cancel{bd}$$

How many multiplications do we need?

$$= (ad + bc)$$

Only 3! If we do extra additions and subtractions.
Compute $\underline{ac}, \underline{bd}, \underline{(a + b)(c + d)}$. Then
$(ad + bc) = (a + b)(c + d) - ac - bd$

Improving the Running Time

$$\begin{aligned} xy &= (10^{n/2}x_L + x_R)(10^{n/2}y_L + y_R) \\ &= 10^n \underbrace{x_L y_L}_{n/2} + 10^{n/2}\underbrace{(x_L y_R + x_R y_L)} + \underbrace{x_R y_R}_{n/2} \end{aligned}$$

Gauss trick: $x_L y_R + x_R y_L = \underbrace{(x_L + x_R)}_{\approx n/2}\underset{\approx n/2}{(y_L + y_R)} - x_L y_L - x_R y_R$

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Gauss trick: $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$

Recursively compute only $x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R)$.

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Time Analysis

Running time is given by

$$T(n) = 3T(n/2) + O(n) \qquad T(1) = O(1)$$

which means

Improving the Running Time

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Recursively compute only $x_L y_L, x_R y_R, (x_L + x_R)(y_L + y_R)$.

Time Analysis

Running time is given by

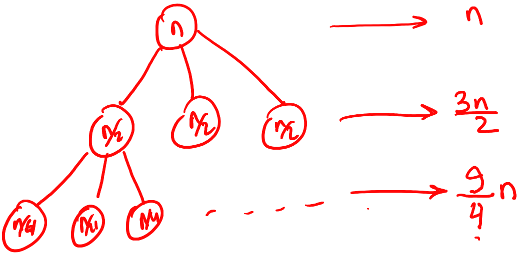
$$T(n) = 3T(n/2) + O(n) \qquad T(1) = O(1)$$

which means $T(n) = O(n^{\log_2 3}) = O(n^{1.585})$

Analyzing the Recurrences

1. Basic divide and conquer: $T(n) = 4T(n/2) + O(n)$, $T(1) = 1$. **Claim:** $T(n) = \Theta(n^2)$.
2. Saving a multiplication: $T(n) = 3T(n/2) + O(n)$, $T(1) = 1$. **Claim:** $T(n) = \Theta(n^{1+\log 1.5})$

$a^{\log_b n} = n^{\log_b a}$



$\left(\frac{3}{2}\right)^{i} n \leftarrow$

$$\underbrace{\left(\frac{3}{2}\right)^{\log_2 n} n}_{\uparrow} = n^{\log_2(3/2)} \times n = n^{1+\log(1.5)}$$

$$= n^{\log_2(3) - \log_2 2} \times n = n^{\log_2(3)} << n^2$$

Analyzing the Recurrences

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Use recursion tree method:

1. In both cases, depth of recursion $L = \log n$.
2. Work at depth i is $4^i n/2^i$ and $3^i n/2^i$ respectively: number of children at depth i times the work at each child
3. Total work is therefore $n\sum_{i=0}^{L} 2^i$ and $n\sum_{i=0}^{L}(3/2)^i$ respectively.

State of the Art

Schönhage-Strassen 1971: $O(n \log n \log \log n)$ time using Fast-Fourier-Transform (FFT)

Martin Fürer 2007: $O(n \log n 2^{O(\log^* n)})$ time

Conjecture

There is an $O(n \log n)$ time algorithm.

Part II

Selecting in Unsorted Lists

Rank of element in an array

A: an unsorted array of n integers

Definition

For $1 \leq j \leq n$, element of rank j is the j'th smallest element in A.

Unsorted array	16	14	34	20	12	5	3	19	11
Ranks	6	5	9	8	4	2	1	7	3
Sort of array	3	5	11	12	14	16	19	20	34

Problem - Selection

Input	Unsorted array A of n integers **and** integer j
Goal	Find the jth smallest number in A (*rank* j number)

Median: $j = \lfloor (n+1)/2 \rfloor$

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Simplifying assumption for sake of notation: elements of A are distinct

Algorithm I

1. Sort the elements in A
2. Pick jth element in sorted order

Time taken $= O(n \log n)$

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Do we need to sort? Is there an $O(n)$ time algorithm?

Algorithm II

If j is small or $n - j$ is small then

1. Find j smallest/largest elements in A in $O(jn)$ time. (How?)

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1. Find j smallest/largest elements in A in $O(jn)$ time. (How?)
2. Time to find median is $O(n^2)$.

QuickSelect

Divide and Conquer Approach

1. Pick a pivot element a from A
2. Partition A based on a.
   $A_{\text{less}} = \{x \in A \mid x \leq a\}$ and $A_{\text{greater}} = \{x \in A \mid x > a\}$

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4. $|A_{\text{less}}| > j$: recursively find jth smallest element in A_{less}

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3. $|A_{\text{less}}| = j$: return a
4. $|A_{\text{less}}| > j$: recursively find jth smallest element in A_{less}
5. $|A_{\text{less}}| < j$: recursively find kth smallest element in A_{greater} where $k = j - |A_{\text{less}}|$.

Example

$j = 5$

16	14	34	20	12	5	3	19	11

$\underbrace{14\ 12\ 5\ 3\ 11\ 16}\ \underbrace{34\ 20\ 19}_{A_{greater}}$

$|A_{less}| = 6$

$\underbrace{3\ 5\ 11\ 12\ 14}_{|A_{less}| = 5}\ \underbrace{16}_{A_{gr}}$

14

Time Analysis

1. Partitioning step: $O(n)$ time to scan A
2. How do we choose pivot? Recursive running time?

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Suppose we always choose pivot to be $A[1]$.

Say A is sorted in increasing order and $j = n$.
Exercise: show that algorithm takes $\Omega(n^2)$ time

A Better Pivot

Suppose pivot is the ℓth smallest element where $n/4 \leq \ell \leq 3n/4$. That is pivot is *approximately* in the middle of A

Then $n/4 \leq |A_{\text{less}}| \leq 3n/4$ *and* $n/4 \leq |A_{\text{greater}}| \leq 3n/4$. If we apply recursion,

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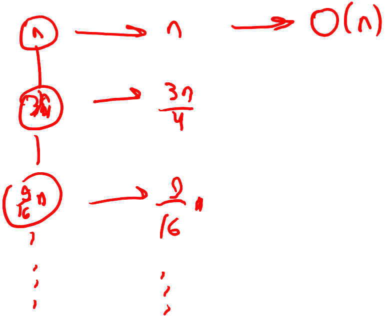
$$T(n) \leq T(3n/4) + O(n)$$

Implies $T(n) = O(n)$!



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Analysis a little bit later.

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Implies $T(n) = O(n)$!

How do we find such a pivot? Randomly? In fact works!
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Can we choose pivot deterministically?

Divide and Conquer Approach

A game of medians

Idea

1. Break input A into many subarrays: $L_1, \ldots L_k$.
2. Find median m_i in each subarray L_i.
3. Find the median x of the medians $m_1, \ldots, m_k$.
4. Intuition: The median x should be close to being a good median of all the numbers in A.
5. Use x as pivot in previous algorithm.

New example

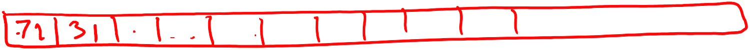
$n/5$

The input:

75	31	13	26	83	110	60	120	63	30	3	41	44	107	30	23	91	17	6	110
68	24	41	26	58	57	61	20	52	45	13	79	86	91	55	66	13	103	36	70
19	40	45	111	56	74	17	95	96	77	29	65	36	96	93	119	9	61	3	9
100	3	88	47	115	107	79	39	109	20	59	25	92	81	36	10	30	113	73	116
72	58	24	16	12	69	40	24	19	92	7	65	75	41	43	117	103	38	8	20

n

72 31 . - - - - - - - median



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Compute median of the medians (recursive call):

72	74	13	66
31	60	65	30
41	39	75	61
26	63	91	8
58	45	43	70

↑ ↑ ↑ ↑

$\underbrace{41 \; 60 \; 65 \; 61} = 60$

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After partition (pivot **60**):

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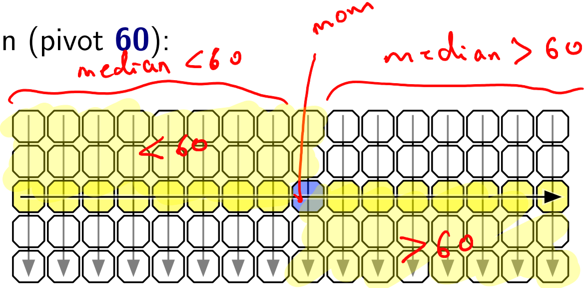

Choosing the pivot

A clash of medians

1. Partition array A into $\lceil n/5 \rceil$ lists of 5 items each.
   $L_1 = \{A[1], A[2], \ldots, A[5]\}$, $L_2 = \{A[6], \ldots, A[10]\}$, $\ldots$,
   $L_i = \{A[5i+1], \ldots, A[5i-4]\}$, $\ldots$,
   $L_{\lceil n/5 \rceil} = \{A[5\lceil n/5 \rceil - 4, \ldots, A[n]\}$.
2. For each i find median b_i of L_i using brute-force in $O(1)$ time. Total $O(n)$ time
3. Let $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
4. Find median b of B

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   $L_{\lceil n/5 \rceil} = \{A[5\lceil n/5 \rceil - 4, \ldots, A[n]\}$.
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3. Let $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
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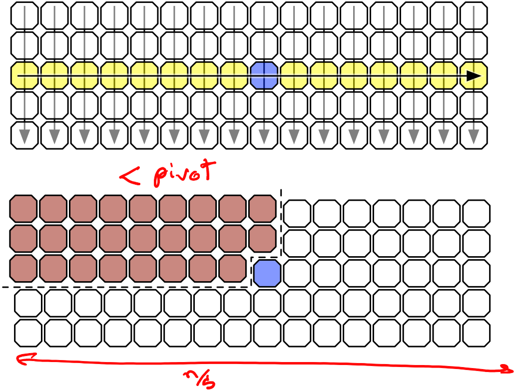
Lemma

Median of B is an approximate *median of A. That is, if b is used a pivot to partition A, then $|A_{less}| \leq 7n/10 + 6$* and *$|A_{greater}| \leq 7n/10 + 6$.*

Median of Medians: Proof of Lemma

Proposition

There are at least $3n/10 - 6$ *elements smaller than the median of medians* b.



$3\left(\frac{n}{10}+1\right)-1$

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Proof.

At least half of the $\lfloor n/5 \rfloor$ groups have at least 3 elements smaller than b, except for the group containing b which has 2 elements smaller than b. Hence number of elements smaller than b is:

$$3\lfloor \frac{\lfloor n/5 \rfloor + 1}{2} \rfloor - 1 \geq 3n/10 - 6 \qquad \square$$

Median of Medians: Proof of Lemma

Proposition

There are at least $3n/10 - 6$ *elements smaller than the median of medians* b.

Corollary

$|A_{greater}| \leq 7n/10 + 6$.

Via symmetric argument,

Corollary

$|A_{less}| \leq 7n/10 + 6$.

Algorithm for Selection

A storm of medians

```
select($A$, $j$):
        Form lists $L_1, L_2, \ldots, L_{\lceil n/5 \rceil}$ where $L_i = \{A[5i-4], \ldots, A[5i]\}$
        Find median $b_i$ of each $L_i$ using brute-force
!? →    Find median $b$ of $B = \{b_1, b_2, \ldots, b_{\lceil n/5 \rceil}\}$
        Partition $A$ into $A_{\text{less}}$ and $A_{\text{greater}}$ using $b$ as pivot
        if ($|A_{\text{less}}|) = j$ return $b$
        else if ($|A_{\text{less}}|) > j$)
    →           return select($A_{\text{less}}$, $j$)
        else
    →           return select($A_{\text{greater}}$, $j - |A_{\text{less}}|$)
```

Algorithm for Selection

A storm of medians

```
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```

How do we find median of B?

Algorithm for Selection

A storm of medians

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```

How do we find median of B? Recursively!

Algorithm for Selection

A storm of medians

```
select(A, j):
    Form lists $L_1, L_2, \ldots, L_{\lceil n/5 \rceil}$ where $L_i = \{A[5i-4], \ldots, A[5i]\}$
    Find median $b_i$ of each $L_i$ using brute-force
    $B = [b_1, b_2, \ldots, b_{\lceil n/5 \rceil}]$
    $b =$ select(B, $\lceil n/10 \rceil$)
    Partition A into $A_{\mathrm{less}}$ and $A_{\mathrm{greater}}$ using b as pivot
    if ($|A_{\mathrm{less}}|) = j$ return b
    else if ($|A_{\mathrm{less}}|) > j$)
        return select($A_{\mathrm{less}}$, j)
    else
        return select($A_{\mathrm{greater}}$, $j - |A_{\mathrm{less}}|$)
```

Running time of deterministic median selection

A dance with recurrences

$\leq \frac{7n}{10} + 6$

$$T(n) \leq T(\lceil n/5 \rceil) + \max\{ T(|\underline{A_{\text{less}}}|), T(|\underline{A_{\text{greater}}})|\} + O(n)$$

Running time of deterministic median selection

A dance with recurrences

$$T(n) \leq T(\lceil n/5 \rceil) + \max\{T(|A_{\text{less}}|), T(|A_{\text{greater}})|\} + O(n)$$

From Lemma,

$$T(n) \leq T(\lceil n/5 \rceil) + T(\lfloor 7n/10 + 6 \rfloor) + O(n)$$

and

$$T(n) = O(1) \qquad n < 10$$

Running time of deterministic median selection

A dance with recurrences

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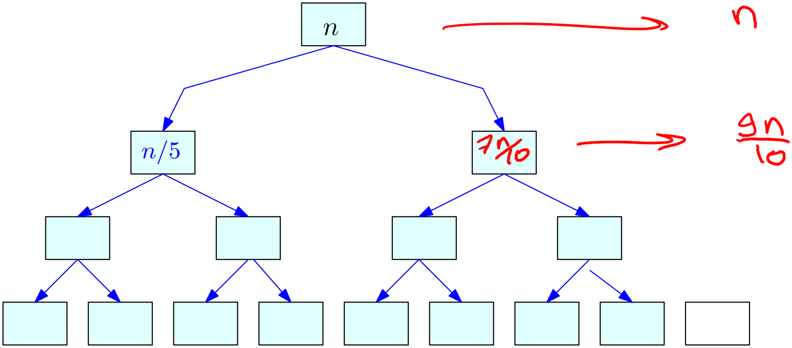
$$T(n) \leq T(\lceil n/5 \rceil) + T(\lfloor 7n/10 + 6 \rfloor) + O(n)$$

and

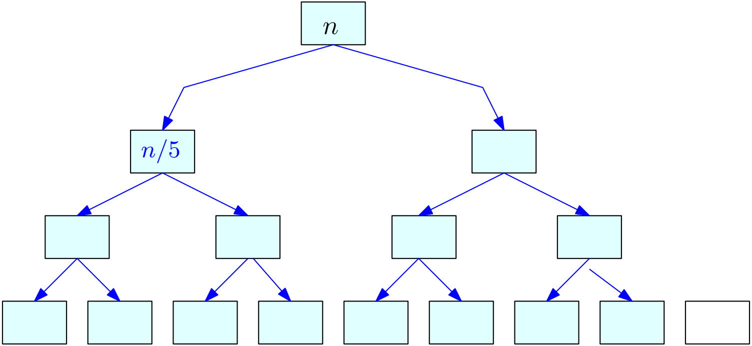
$$T(n) = O(1) \qquad n < 10$$

Exercise: show that $T(n) = O(n)$

Recursion tree fill in

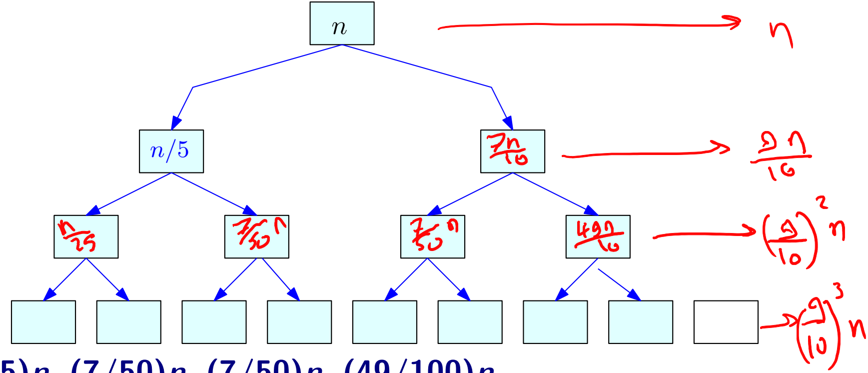


Recursion tree fill in



$(1/5)n$, $(7/10)n$

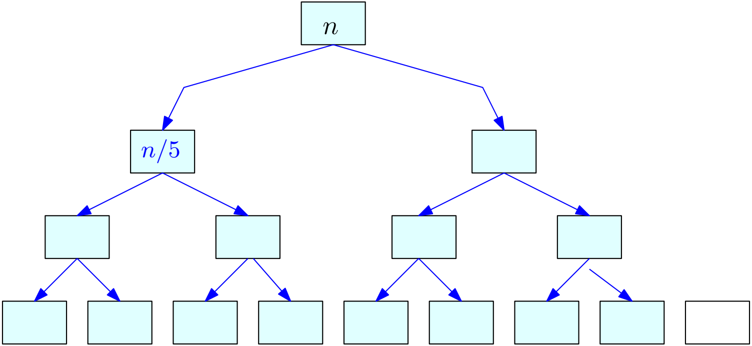
Recursion tree fill in



$(1/25)n$, $(7/50)n$, $(7/50)n$, $(49/100)n$

$\Rightarrow O(n)$

Recursion tree fill in



(1/125)*n*, (7/250)*n*, (7/250)*n*, (49/500)*n*, (7/250)*n*, (49/500)*n*, (49/500)*n*, (343/1000)*n*

Summary: Selection in linear time

Theorem

The algorithm **select**$(A[1 \,..\, n], k)$ *computes in* $O(n)$ *deterministic time the* k*th smallest element in* A*.*

On the other hand, we have:

Lemma

The algorithm **QuickSelect**$(A[1 \,..\, n], k)$ *computes the* k*th smallest element in* A*. The running time of* **QuickSelect** *is* $\Theta(n^2)$ *in the worst case.*

Questions to ponder

1. Why did we choose lists of size $\mathbf{5}$? Will lists of size $\mathbf{3}$ work?
2. Write a recurrence to analyze the algorithm's running time if we choose a list of size k.

Median of Medians Algorithm

Due to:
M. Blum, R. Floyd, D. Knuth, V. Pratt, R. Rivest, and R. Tarjan.
“Time bounds for selection”.
Journal of Computer System Sciences (JCSS), 1973.

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How many Turing Award winners in the author list?
All except Vaughn Pratt!

Takeaway Points

1. Recursion tree method and guess and verify are the most reliable methods to analyze recursions in algorithms.
2. Recursive algorithms naturally lead to recurrences.
3. Some times one can look for certain type of recursive algorithms (reverse engineering) by understanding recurrences and their behavior.