

# Depth First Search (DFS)

## Lecture 16

Thursday, March 14, 2019

LaTeXed: March 12, 2019 16:20

# Today

Two topics:

- Structure of directed graphs
- **DFS** and its properties
- One application of **DFS** to obtain fast algorithms

# Part I

## Strong connected components

# Strong Connected Components (SCCs)

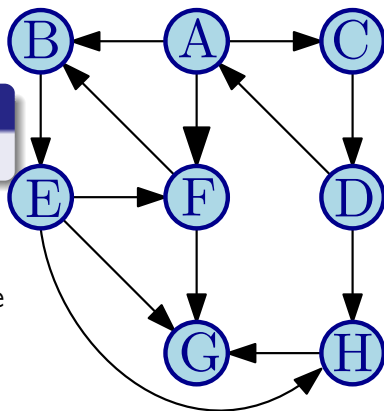
## Algorithmic Problem

Find all **SCCs** of a given directed graph.

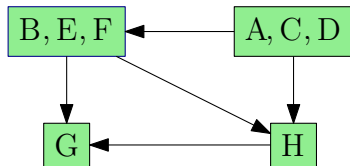
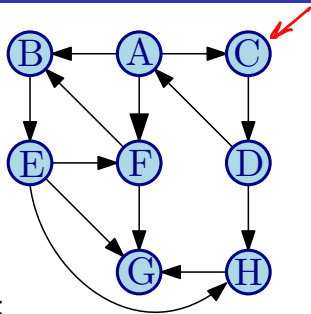
Previous lecture:

Saw an  $O(n \cdot (n + m))$  time algorithm.

This lecture: sketch of a  $O(n + m)$  time algorithm.



# Graph of SCCs



Graph of SCCs  $G^{\text{SCC}}$

## Meta-graph of SCCs

Let  $S_1, S_2, \dots, S_k$  be the strong connected components (i.e., SCCs) of  $G$ . The graph of SCCs is  $G^{\text{SCC}}$

- 1 Vertices are  $S_1, S_2, \dots, S_k$
- 2 There is an edge  $(S_i, S_j)$  if there is some  $u \in S_i$  and  $v \in S_j$  such that  $(u, v)$  is an edge in  $G$ .

# Reversal and SCCs

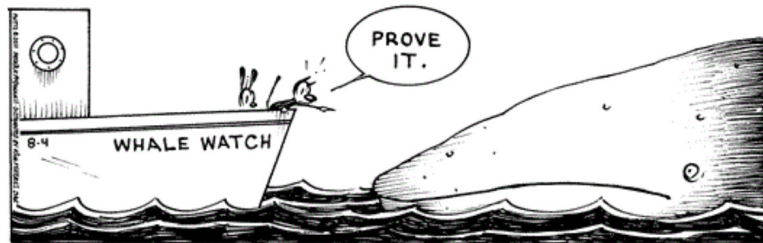
## Proposition

For any graph  $G$ , the graph of SCCs of  $G^{\text{rev}}$  is the same as the reversal of  $G^{\text{SCC}}$ .

## Proof.

Exercise.

MUTTS by Patrick McDonnell | 08/04/11



# SCCs and DAGs

## Proposition

For any graph  $G$ , the graph  $G^{\text{SCC}}$  has no directed cycle.

## Proof.

If  $G^{\text{SCC}}$  has a cycle  $S_1, S_2, \dots, S_k$  then  $S_1 \cup S_2 \cup \dots \cup S_k$  should be in the same SCC in  $G$ . Formal details: exercise.  $\square$

# Part II

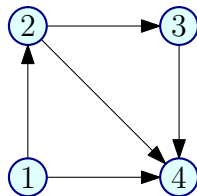
## Directed Acyclic Graphs



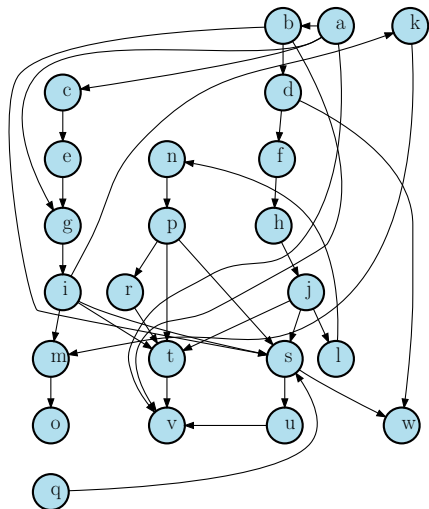
# Directed Acyclic Graphs

## Definition

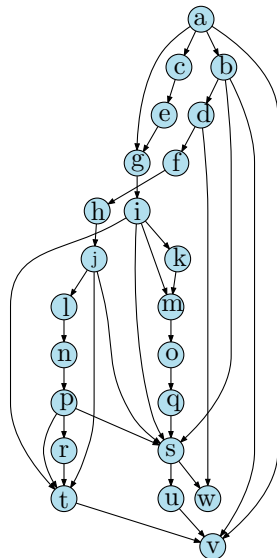
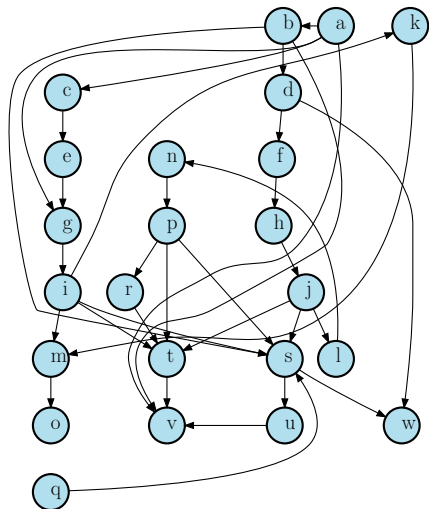
A directed graph  $G$  is a **directed acyclic graph** (**DAG**) if there is no directed cycle in  $G$ .



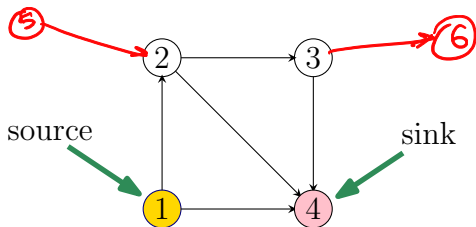
# Is this a DAG?



# Is this a DAG?



# Sources and Sinks



## Definition

- 1 A vertex  $u$  is a **source** if it has no in-coming edges.
- 2 A vertex  $u$  is a **sink** if it has no out-going edges.

# Simple DAG Properties

## Proposition

*Every DAG  $G$  has at least one source and at least one sink.*

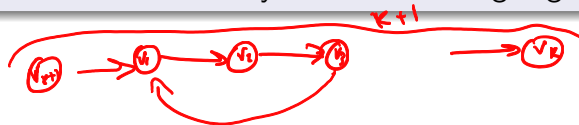
# Simple DAG Properties

## Proposition

Every DAG  $G$  has at least one source and at least one sink.

## Proof.

Let  $P = v_1, v_2, \dots, v_k$  be a longest path in  $G$ . Claim that  $v_1$  is a source and  $v_k$  is a sink. Suppose not. Then  $v_1$  has an incoming edge which either creates a cycle or a longer path both of which are contradictions. Similarly if  $v_k$  has an outgoing edge.



# Simple DAG Properties

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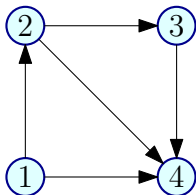
①  $G$  is a DAG if and only if  $G^{\text{rev}}$  is a DAG.

②  $G$  is a DAG if and only if each node is in its own strong connected component.

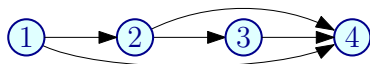


Formal proofs: exercise.

# Topological Ordering/Sorting



Graph  $G$



Topological Ordering of  $G$

## Definition

A **topological ordering/topological sorting** of  $G = (V, E)$  is an ordering  $\prec$  on  $V$  such that if  $(u, v) \in E$  then  $u \prec v$ .

## Informal equivalent definition:

One can order the vertices of the graph along a line (say the  $x$ -axis) such that all edges are from left to right.



# DAGs and Topological Sort

## Lemma

*A directed graph  $G$  can be topologically ordered iff it is a DAG.*

Need to show both directions.

# DAGs and Topological Sort

## Lemma

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# DAGs and Topological Sort

## Lemma

A directed graph  $G$  can be topologically ordered if it is a DAG.

## Proof.

Consider the following algorithm:

- 1 Pick a source  $u$ , output it.
- 2 Remove  $u$  and all edges out of  $u$ .
- 3 Repeat until graph is empty.

Exercise: prove this gives topological sort. □

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## Lemma

*A directed graph  $G$  can be topologically ordered if it is a DAG.*

## Proof.

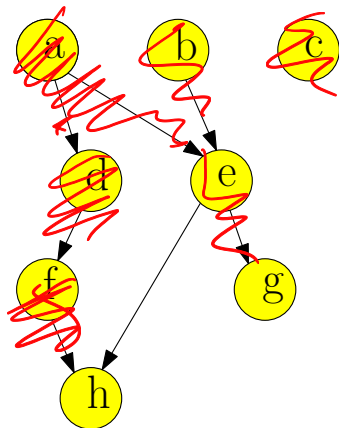
Consider the following algorithm:

- 1 Pick a source  $u$ , output it.
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Exercise: prove this gives topological sort. □

Exercise: show algorithm can be implemented in  $O(m + n)$  time.

# Topological Sort: Example



# DAGs and Topological Sort

## Lemma

*A directed graph  $G$  can be topologically ordered only if it is a DAG.*

# DAGs and Topological Sort

## Lemma

A directed graph  $G$  can be topologically ordered only if it is a **DAG**.

## Proof.

Suppose  $G$  is not a **DAG** and has a topological ordering  $\prec$ .  $G$  has a cycle  $C = u_1, u_2, \dots, u_k, u_1$ .

Then  $u_1 \prec u_2 \prec \dots \prec u_k \prec u_1$ !

That is...  $u_1 \prec u_1$ .

A contradiction (to  $\prec$  being an order).

Not possible to topologically order the vertices. □

# DAGs and Topological Sort

**Note:** A DAG  $G$  may have many different topological sorts.

**Question:** What is a DAG with the most number of distinct topological sorts for a given number  $n$  of vertices?

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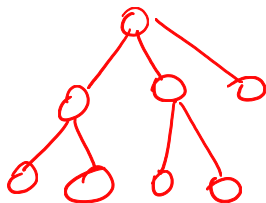




# Cycles in graphs

→ **Question:** Given an *undirected* graph how do we check whether it has a cycle and output one if it has one?

→ **Question:** Given an *directed* graph how do we check whether it has a cycle and output one if it has one?



# To Remember: Structure of Graphs

**Undirected graph:** connected components of  $G = (V, E)$  partition  $V$  and can be computed in  $O(m + n)$  time.

**Directed graph:** the meta-graph  $G^{\text{SCC}}$  of  $G$  can be computed in  $O(m + n)$  time.  $G^{\text{SCC}}$  gives information on the partition of  $V$  into strong connected components and how they form a DAG structure.

Above structural decomposition will be useful in several algorithms

## Part III

# Depth First Search (DFS)

# Depth First Search

- ① **DFS** special case of Basic Search.
- ② **DFS** is useful in understanding graph structure.
- ③ **DFS** used to obtain linear time ( $O(m + n)$ ) algorithms for
  - ① Finding cut-edges and cut-vertices of undirected graphs
  - ② Finding strong connected components of directed graphs
  - ③ Linear time algorithm for testing whether a graph is planar
- ④ ...many other applications as well.

# DFS in Undirected Graphs

Recursive version. Easier to understand some properties.

**DFS( $G$ )**

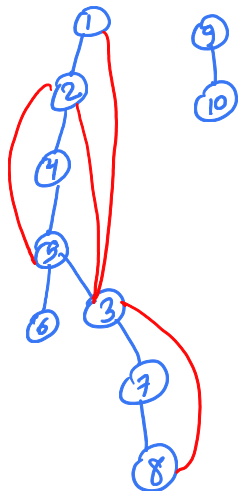
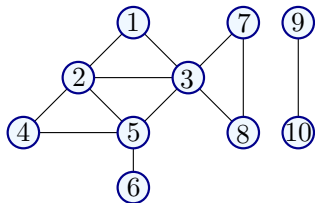
```
for all  $u \in V(G)$  do
  Mark  $u$  as unvisited
  Set  $\text{pred}(u)$  to null
 $T$  is set to  $\emptyset$ 
while  $\exists$  unvisited  $u$  do
  DFS( $u$ )
Output  $T$ 
```

**DFS( $u$ )**

```
→ Mark  $u$  as visited
→ for each  $uv$  in  $\text{Out}(u)$  do
  → if  $v$  is not visited then
    → add edge  $uv$  to  $T$ 
    → set  $\text{pred}(v)$  to  $u$ 
    → DFS( $v$ )
```

Implemented using a global array *Visited* for all recursive calls.  
 $T$  is the search tree/forest.

# Example



Edges classified into two types:  $uv \in E$  is a

- 1 **tree edge**: belongs to  $T$
- 2 **non-tree edge**: does not belong to  $T$

# Properties of DFS tree

## Proposition

- ①  $T$  is a forest
- ② connected components of  $T$  are same as those of  $G$ .
- ③ If  $uv \in E$  is a non-tree edge then, in  $T$ , either:
  - ①  $u$  is an ancestor of  $v$ , or
  - ②  $v$  is an ancestor of  $u$ .

**Question:** Why are there no *cross-edges*?

# DFS with Visit Times

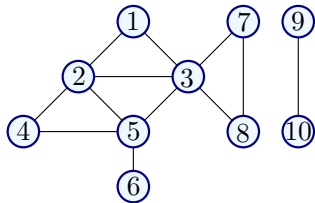
Keep track of when nodes are visited.

```
DFS( $G$ )  
  for all  $u \in V(G)$  do  
    Mark  $u$  as unvisited  
   $T$  is set to  $\emptyset$   
  →  $time = 0$   
  while  $\exists$  unvisited  $u$  do  
    DFS( $u$ )  
  Output  $T$ 
```

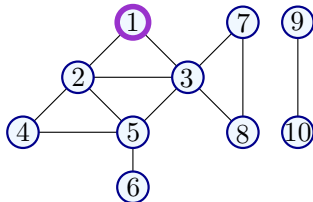
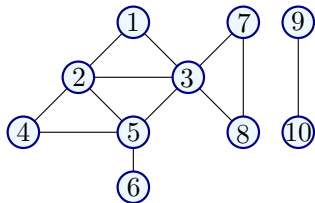
```
DFS( $u$ )  
  Mark  $u$  as visited  
   $pre(u) = ++time$   
  for each  $uv$  in  $Out(u)$  do  
    if  $v$  is not marked then  
      add edge  $uv$  to  $T$   
      DFS( $v$ )  
   $post(u) = ++time$ 
```



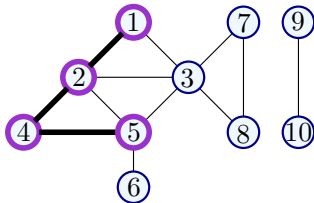
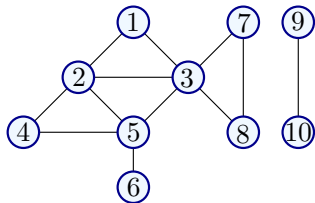
# Example



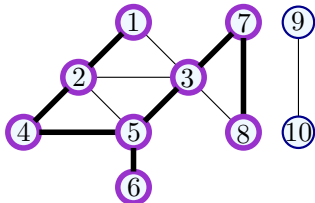
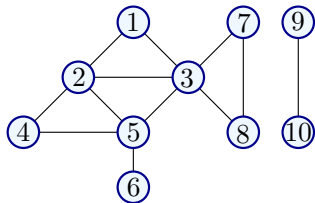
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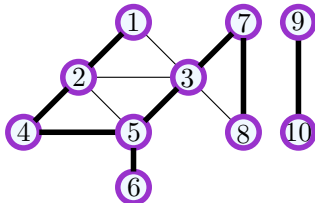
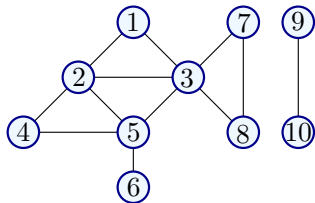
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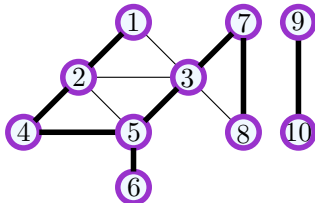
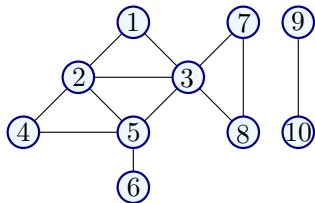
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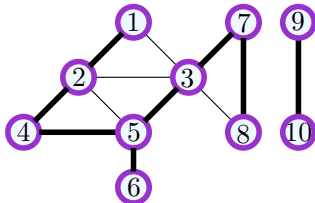
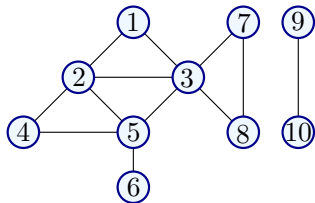


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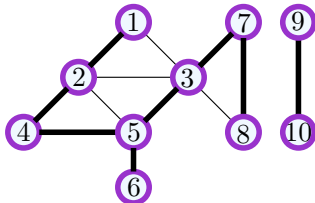
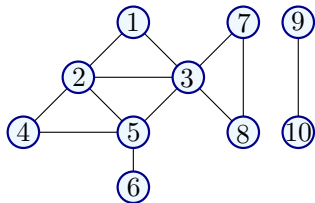
vertex	$[pre, post]$
1	$[1, ]$

# Example



vertex	$[pre, post]$
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2	$[2, ]$

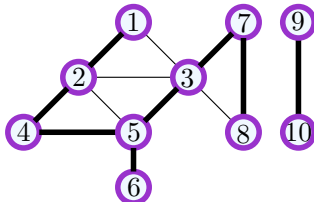
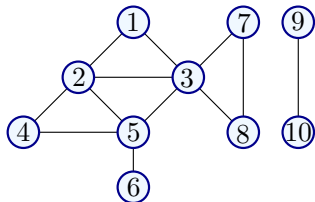
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vertex	$[pre, post]$
1	$[1, ]$
2	$[2, ]$
4	$[3, ]$

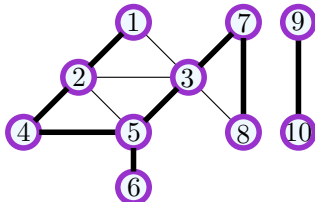
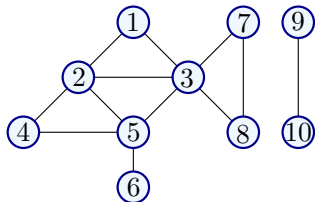


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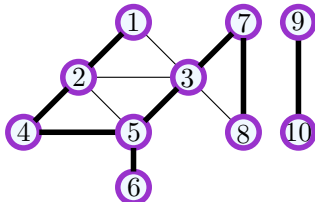
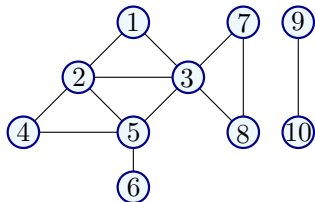
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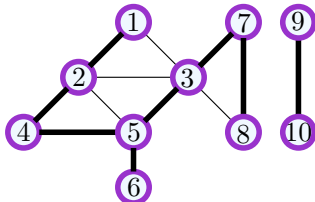
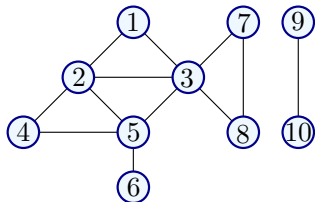
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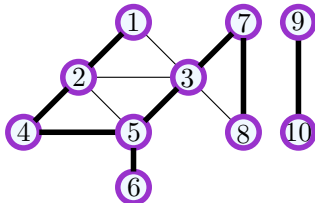
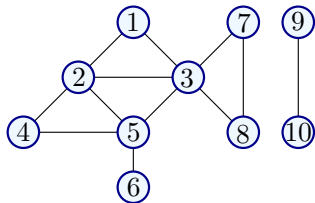
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5	$[4, ]$
6	$[5, 6]$

# Example



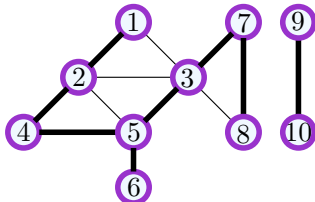
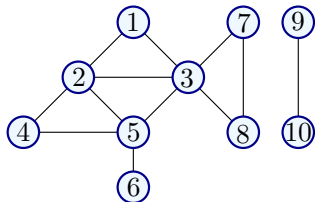
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5	$[4, ]$
6	$[5, 6]$
3	$[7, ]$

# Example



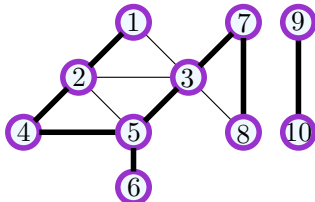
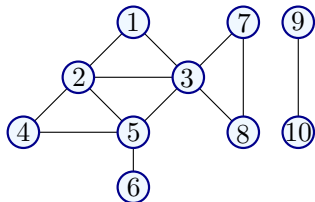
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7	[8, ]

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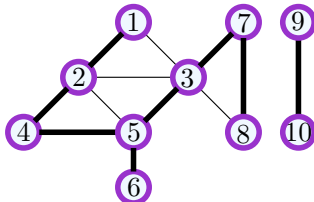
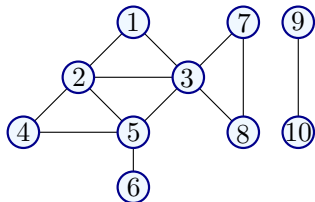
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6	[5, 6]
3	[7, ]
7	[8, ]
8	[9, ]

# Example



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6	[5, 6]
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7	[8, ]
8	[9, 10]

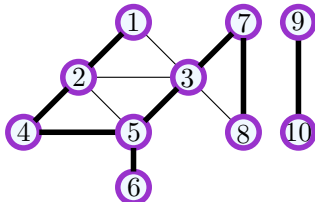
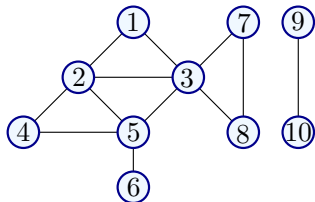
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7	$[8, 11]$
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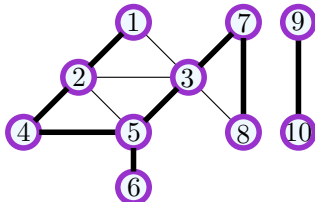
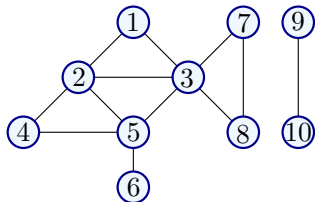


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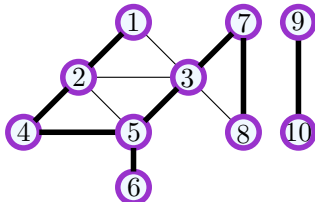
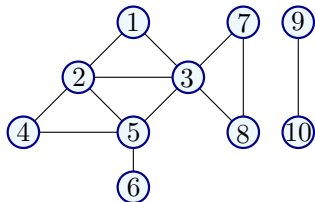
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4	$[3, ]$
5	$[4, ]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$

# Example



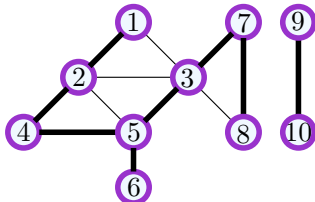
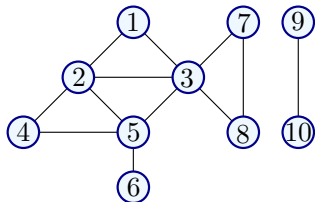
vertex	$[pre, post]$
1	$[1, ]$
2	$[2, ]$
4	$[3, ]$
5	$[4, 13]$
6	$[5, 6]$
3	$[7, 12]$
7	$[8, 11]$
8	$[9, 10]$

# Example



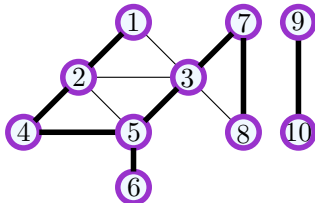
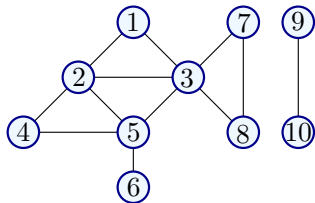
vertex	$[pre, post]$
1	$[1, ]$
2	$[2, ]$
4	$[3, 14]$
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7	$[8, 11]$
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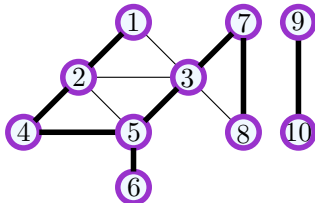
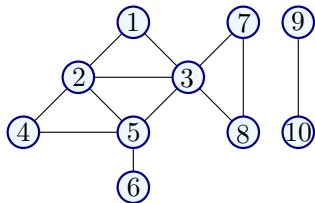
vertex	$[pre, post]$
1	$[1, ]$
2	$[2, 15]$
4	$[3, 14]$
5	$[4, 13]$
6	$[5, 6]$
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# Example



vertex	$[pre, post]$
1	[1, 16]
2	[2, 15]
4	[3, 14]
5	[4, 13]
6	[5, 6]
3	[7, 12]
7	[8, 11]
8	[9, 10]

# Example



vertex	<i>[pre, post]</i>
1	[1, 16]
2	[2, 15]
4	[3, 14]
5	[4, 13]
6	[5, 6]
3	[7, 12]
7	[8, 11]
8	[9, 10]
9	[17, 20]
10	[18, 19]

# pre and post numbers

Node  $u$  is **active** in time interval  $[\text{pre}(u), \text{post}(u)]$

## Proposition

*For any two nodes  $u$  and  $v$ , the two intervals  $[\text{pre}(u), \text{post}(u)]$  and  $[\text{pre}(v), \text{post}(v)]$  are disjoint or one is contained in the other.*

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**pre** and **post** numbers useful in several applications of **DFS**

# DFS in Directed Graphs

## DFS( $G$ )

Mark all nodes  $u$  as unvisited

$T$  is set to  $\emptyset$

$time = 0$

**while** there is an unvisited node  $u$  **do**

    DFS( $u$ )

Output  $T$

## DFS( $u$ )

Mark  $u$  as visited

pre( $u$ ) = ++ $time$

**for** each edge  $(u, v)$  in  $Out(u)$  **do**

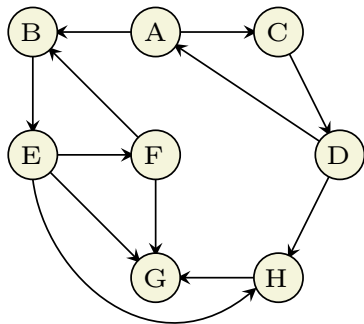
**if**  $v$  is not visited

        add edge  $(u, v)$  to  $T$

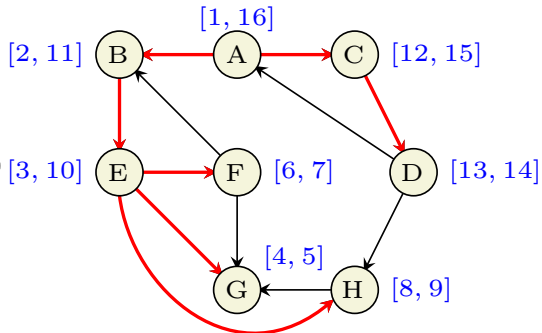
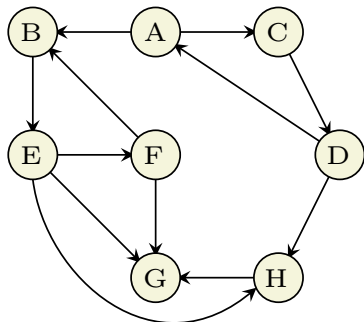
        DFS( $v$ )

post( $u$ ) = ++ $time$

# Example



# Example



# DFS Properties

Generalizing ideas from undirected graphs:

- 1 **DFS( $G$ )** takes  $O(m + n)$  time.



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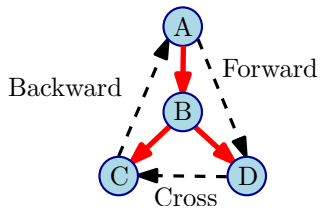
**Note:** Not obvious whether **DFS( $G$ )** is useful in directed graphs but it is.

# DFS Tree

Edges of  $G$  can be classified with respect to the **DFS** tree  $T$  as:

- 1 **Tree edges** that belong to  $T$
- 2 A **forward edge** is a non-tree edges  $(x, y)$  such that  $\text{pre}(x) < \text{pre}(y) < \text{post}(y) < \text{post}(x)$ .
- 3 A **backward edge** is a non-tree edge  $(y, x)$  such that  $\text{pre}(x) < \text{pre}(y) < \text{post}(y) < \text{post}(x)$ .
- 4 A **cross edge** is a non-tree edges  $(x, y)$  such that the intervals  $[\text{pre}(x), \text{post}(x)]$  and  $[\text{pre}(y), \text{post}(y)]$  are disjoint.

# Types of Edges



# Cycles in graphs

**Question:** Given an *undirected* graph how do we check whether it has a cycle and output one if it has one?

**Question:** Given an *directed* graph how do we check whether it has a cycle and output one if it has one?

# Using DFS...

... to check for Acyclicity and compute Topological Ordering

## Question

Given  $G$ , is it a **DAG**? If it is, generate a topological sort. Else output a cycle  $C$ .



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**DFS** based algorithm:

- 1 Compute **DFS**( $G$ )
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- 3 Otherwise output nodes in decreasing post-visit order. **Note**: no need to sort, **DFS**( $G$ ) can output nodes in this order.

Algorithm runs in  $O(n + m)$  time.

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Correctness is not so obvious. See next two propositions.

# Back edge and Cycles

## Proposition

$G$  has a cycle iff there is a back-edge in **DFS**( $G$ ).

## Proof.

If:  $(u, v)$  is a back edge implies there is a cycle  $C$  consisting of the path from  $v$  to  $u$  in **DFS** search tree and the edge  $(u, v)$ .

Only if: Suppose there is a cycle  $C = v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k \rightarrow v_1$ . Let  $v_i$  be first node in  $C$  visited in **DFS**.

All other nodes in  $C$  are descendants of  $v_i$  since they are reachable from  $v_i$ .

Therefore,  $(v_{i-1}, v_i)$  (or  $(v_k, v_1)$  if  $i = 1$ ) is a back edge. □

## Proposition

If  $G$  is a DAG and  $\text{post}(v) > \text{post}(u)$ , then  $(u, v)$  is not in  $G$ .

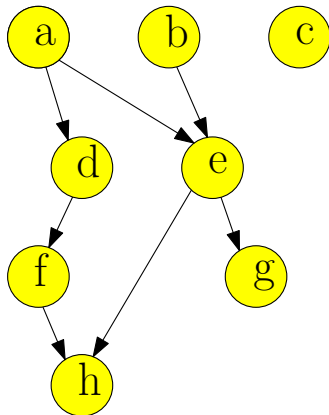
## Proof.

Assume  $\text{post}(v) > \text{post}(u)$  and  $(u, v)$  is an edge in  $G$ . We derive a contradiction. One of two cases holds from DFS property.

- **Case 1:**  $[\text{pre}(u), \text{post}(u)]$  is contained in  $[\text{pre}(v), \text{post}(v)]$ .  
Implies that  $u$  is explored during  $\text{DFS}(v)$  and hence is a descendent of  $v$ . Edge  $(u, v)$  implies a cycle in  $G$  but  $G$  is assumed to be DAG!
- **Case 2:**  $[\text{pre}(u), \text{post}(u)]$  is disjoint from  $[\text{pre}(v), \text{post}(v)]$ .  
This cannot happen since  $v$  would be explored from  $u$ .



# Example



## Part IV

Linear time algorithm for finding all strong connected components of a directed graph

# Finding all SCCs of a Directed Graph

## Problem

Given a directed graph  $G = (V, E)$ , output *all* its strong connected components.

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Straightforward algorithm:

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Mark all vertices in  $V$  as not visited.  
for each vertex  $u \in V$  not visited yet do  
  find  $\text{SCC}(G, u)$  the strong component of  $u$ :  
    Compute  $\text{rch}(G, u)$  using  $\text{DFS}(G, u)$   
    Compute  $\text{rch}(G^{\text{rev}}, u)$  using  $\text{DFS}(G^{\text{rev}}, u)$   
     $\text{SCC}(G, u) \leftarrow \text{rch}(G, u) \cap \text{rch}(G^{\text{rev}}, u)$   
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```

Running time:  $O(n(n + m))$



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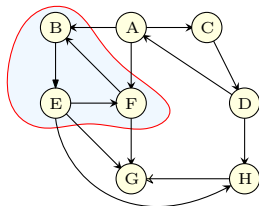
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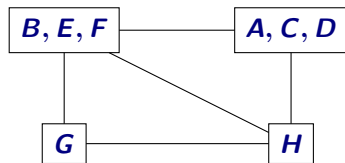
Running time:  $O(n(n + m))$

Is there an  $O(n + m)$  time algorithm?

# Structure of a Directed Graph



Graph  $G$



Graph of **SCCs**  $G^{\text{SCC}}$

## Reminder

$G^{\text{SCC}}$  is created by collapsing every strong connected component to a single vertex.

## Proposition

For a directed graph  $G$ , its meta-graph  $G^{\text{SCC}}$  is a **DAG**.

# Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

## Wishful Thinking Algorithm

- 1 Let  $u$  be a vertex in a *sink* SCC of  $G^{\text{SCC}}$
- 2 Do **DFS**( $u$ ) to compute **SCC**( $u$ )
- 3 Remove **SCC**( $u$ ) and repeat

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- 4 Therefore, total time  $O(n + m)$ !

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How do we find a vertex in a sink **SCC** of  $G^{\text{SCC}}$ ?



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Can we obtain an *implicit* topological sort of  $G^{\text{SCC}}$  without computing  $G^{\text{SCC}}$ ?

**Answer:** **DFS( $G$ )** gives some information!

# Linear Time Algorithm

...for computing the strong connected components in  $G$

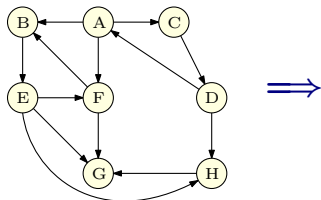
```
do DFS( $G^{\text{rev}}$ ) and output vertices in decreasing post order.  
Mark all nodes as unvisited  
for each  $u$  in the computed order do  
  if  $u$  is not visited then  
    DFS( $u$ )  
    Let  $S_u$  be the nodes reached by  $u$   
    Output  $S_u$  as a strong connected component  
    Remove  $S_u$  from  $G$ 
```

## Theorem

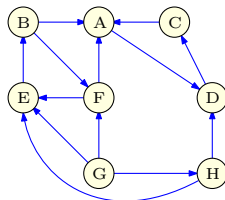
*Algorithm runs in time  $O(m + n)$  and correctly outputs all the SCCs of  $G$ .*

# Linear Time Algorithm: An Example - Initial steps

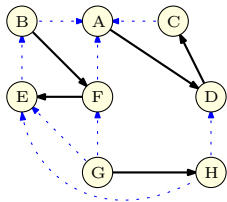
Graph  $G$ :



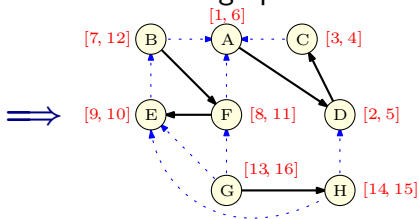
Reverse graph  $G^{\text{rev}}$ :



DFS of reverse graph:



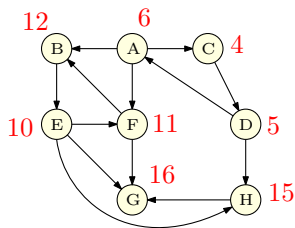
Pre/Post DFS numbering of reverse graph:



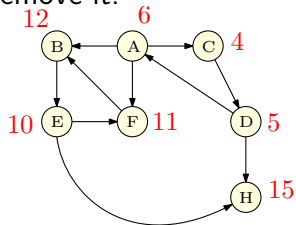
# Linear Time Algorithm: An Example

Removing connected components: 1

Original graph  $G$  with rev post numbers:



Do **DFS** from vertex  $G$   
remove it.

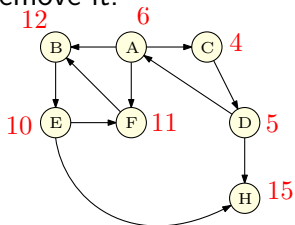


**SCC** computed:  
{ $G$ }

# Linear Time Algorithm: An Example

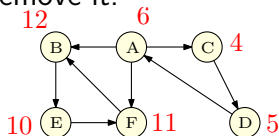
Removing connected components: 2

Do **DFS** from vertex **G**  
remove it.



**SCC** computed:  
{**G**}

Do **DFS** from vertex **H**,  
remove it.

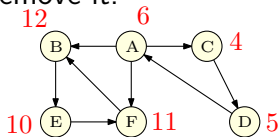


**SCC** computed:  
{**G**}, {**H**}

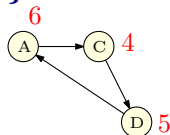
# Linear Time Algorithm: An Example

Removing connected components: 3

Do **DFS** from vertex  $H$ ,  
remove it.



Do **DFS** from vertex  $B$   
Remove visited vertices:  
 $\{F, B, E\}$ .



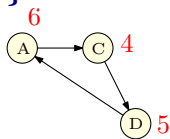
**SCC** computed:  
 $\{G\}, \{H\}$

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 $\{G\}, \{H\}, \{F, B, E\}$

# Linear Time Algorithm: An Example

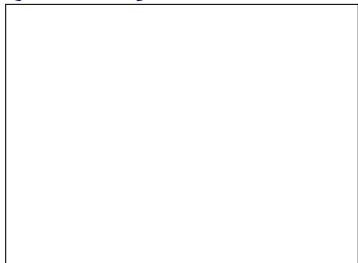
Removing connected components: 4

Do **DFS** from vertex **F**  
Remove visited vertices:  
 $\{F, B, E\}$ .



**SCC** computed:  
 $\{G\}, \{H\}, \{F, B, E\}$

Do **DFS** from vertex **A**  
Remove visited vertices:  
 $\{A, C, D\}$ .

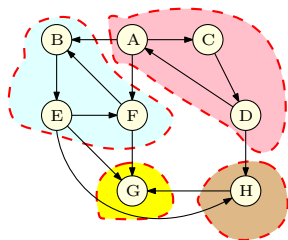


**SCC** computed:  
 $\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$



# Linear Time Algorithm: An Example

Final result



SCC computed:

$\{G\}, \{H\}, \{F, B, E\}, \{A, C, D\}$

Which is the correct answer!

# Obtaining the meta-graph...

Once the strong connected components are computed.

## Exercise:

Given all the strong connected components of a directed graph  $G = (V, E)$  show that the meta-graph  $G^{\text{SCC}}$  can be obtained in  $O(m + n)$  time.

# Solving Problems on Directed Graphs

A template for a class of problems on directed graphs:

- Is the problem solvable when  $G$  is strongly connected?
- Is the problem solvable when  $G$  is a DAG?
- If the above two are feasible then is the problem solvable in a general directed graph  $G$  by considering the meta graph  $G^{\text{SCC}}$ ?

# Part V

An Application to make

# Make/Makefile

- (A) I know what make/makefile is.
- (B) I do NOT know what make/makefile is.

# make Utility [Feldman]

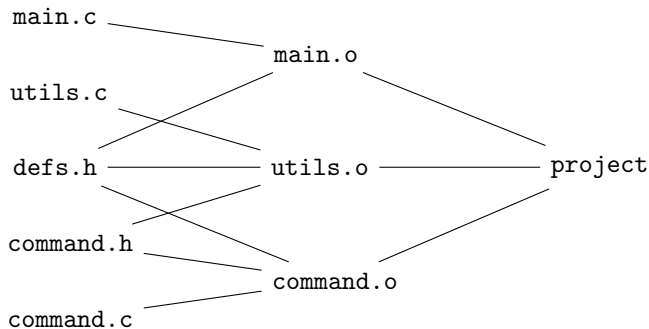
- 1 Unix utility for automatically building large software applications
- 2 A makefile specifies
  - 1 Object files to be created,
  - 2 Source/object files to be used in creation, and
  - 3 How to create them

# An Example makefile

```
project: main.o utils.o command.o
    cc -o project main.o utils.o command.o

main.o: main.c defs.h
    cc -c main.c
utils.o: utils.c defs.h command.h
    cc -c utils.c
command.o: command.c defs.h command.h
    cc -c command.c
```

# makefile as a Digraph





# Computational Problems for `make`

- 1 Is the `makefile` reasonable?
- 2 If it is reasonable, in what order should the object files be created?
- 3 If it is not reasonable, provide helpful debugging information.
- 4 If some file is modified, find the fewest compilations needed to make application consistent.

# Algorithms for make

- 1 Is the makefile reasonable? Is  $G$  a DAG?
- 2 If it is reasonable, in what order should the object files be created? Find a topological sort of a DAG.
- 3 If it is not reasonable, provide helpful debugging information. Output a cycle. More generally, output all strong connected components.
- 4 If some file is modified, find the fewest compilations needed to make application consistent.
  - 1 Find all vertices reachable (using DFS/BFS) from modified files in directed graph, and recompile them in proper order. Verify that one can find the files to recompile and the ordering in linear time.

# Take away Points

- 1 Given a directed graph  $G$ , its **SCCs** and the associated acyclic meta-graph  $G^{\text{SCC}}$  give a structural decomposition of  $G$  that should be kept in mind.
- 2 There is a **DFS** based linear time algorithm to compute all the **SCCs** and the meta-graph. Properties of **DFS** crucial for the algorithm.
- 3 **DAGs** arise in many application and topological sort is a key property in algorithm design. Linear time algorithms to compute a topological sort (there can be many possible orderings so not unique).