

Greedy Algorithms

Lecture 19

Tuesday, March 26, 2019

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Part I

Greedy Algorithms: Tools and Techniques

What is a Greedy Algorithm?

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Greedy algorithms:

- 1 make decision incrementally in small steps *without backtracking*
- 2 decision at each step is based on improving *local or current* state in a myopic fashion without paying attention to the *global* situation
- 3 decisions often based on some fixed and simple *priority* rules

Pros and Cons of Greedy Algorithms

Pros:

- 1 Usually (too) easy to design greedy algorithms
- 2 Easy to implement and often run fast since they are simple
- 3 Several important cases where they are effective/optimal
- 4 Lead to a first-cut heuristic when problem not well understood

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CS 374: Every greedy algorithm needs a proof of correctness

Greedy Algorithm Types

Crude classification:

- 1 **Non-adaptive:** fix some ordering of decisions a priori and stick with the order
- 2 **Adaptive:** make decisions adaptively but greedily/locally at each step

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Plan:

- 1 See several examples
- 2 Pick up some proof techniques

Part II

Scheduling Jobs to Minimize Average Waiting Time

The Problem

- n jobs J_1, J_2, \dots, J_n . J_i has non-negative processing time p_i
- One server/machine/person available to process jobs.
- Schedule/order jobs to min. total or average *waiting time*
- Waiting time of J_i in schedule σ : sum of processing times of all jobs scheduled before J_i

	J_1	J_2	J_3	J_4	J_5	J_6
<i>time</i>	3	4	1	8	2	6

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$$\underline{0} + \underline{3} + \underline{(3 + 4)} + \underline{(3 + 4 + 1)} + (3 + 4 + 1 + 8) + \dots =$$

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Optimal schedule:

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Optimal schedule: Shortest Job First. $J_3, J_5, J_1, J_2, J_6, J_4$.

Optimality of Shortest Job First (SJF)

Theorem

Shortest Job First gives an optimum schedule for the problem of minimizing total waiting time.



$$P_1 \text{ first: } P_1 + P_1 + P_2 = 2P_1 + P_2 \quad \checkmark$$

$$P_2 \text{ first: } P_2 + P_1 + P_2 = P_1 + 2P_2$$

$$P_1 < P_2 \Rightarrow P_1 \text{ first}$$

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Proof strategy: exchange argument

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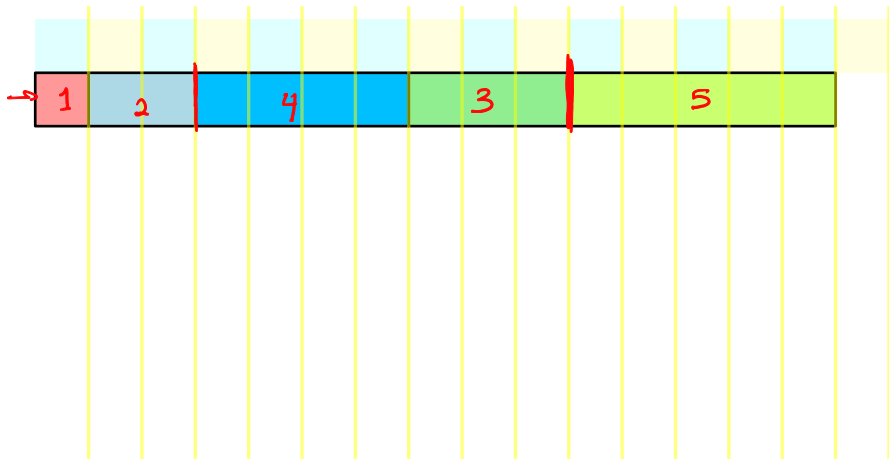
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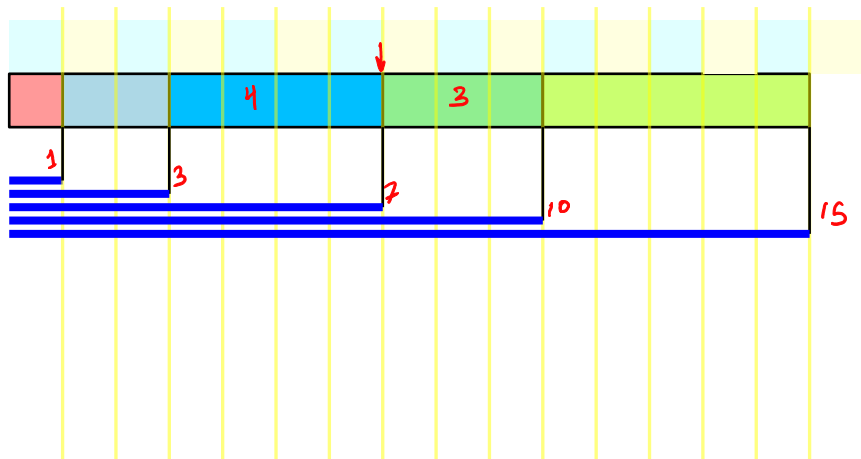
Proof strategy: exchange argument

Assume without loss of generality that job sorted in increasing order of processing time and hence $p_1 \leq p_2 \leq \dots \leq p_n$ and SJF order is J_1, J_2, \dots, J_n .

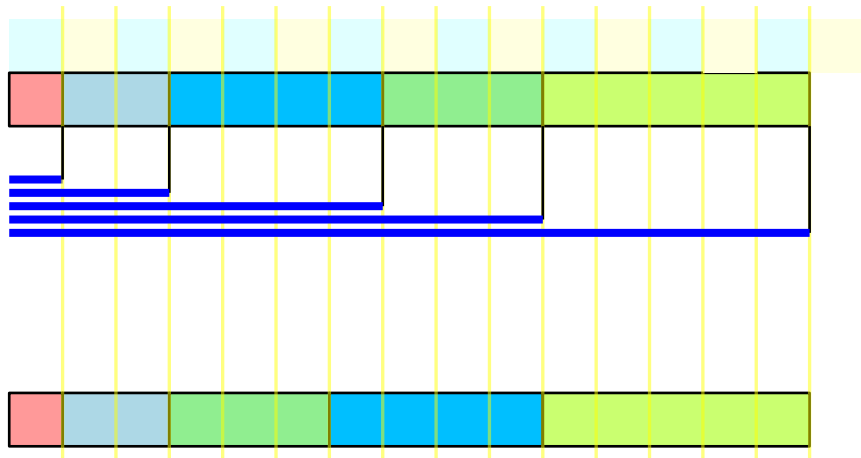
Optimality of SJF: Proof by picture



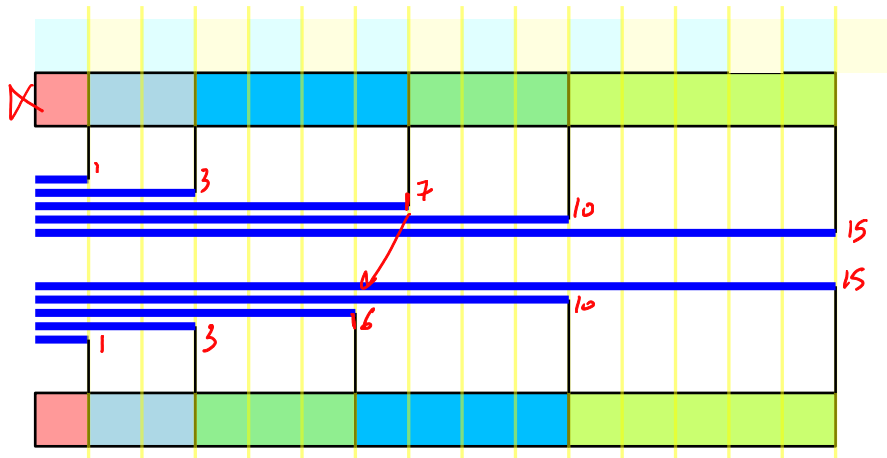
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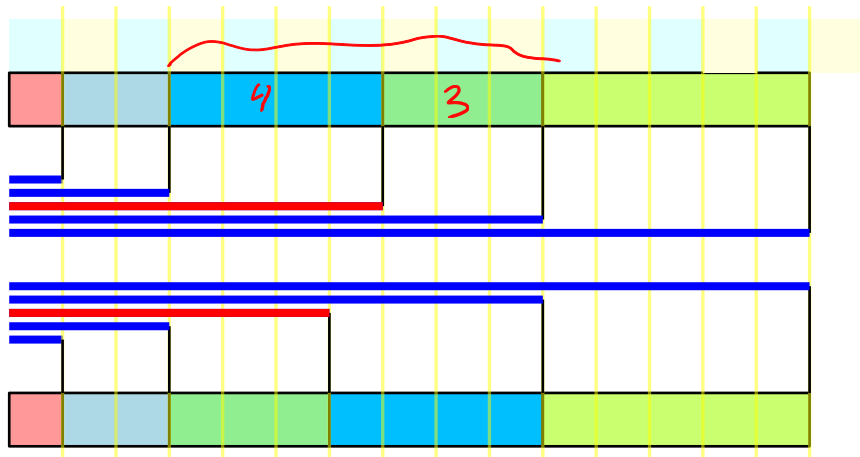
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Inversions

Definition

A schedule $J_{i_1}, J_{i_2}, \dots, J_{i_n}$ has an **inversion** if there are jobs J_a and J_b such that S schedules J_a before J_b , but $p_a > p_b$.

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Claim

If a schedule has an inversion then there is an inversion between two adjacently scheduled jobs.

Proof: exercise.

Proof of optimality of SJF

SJF = Shortest Job First

Recall **SJF** order is J_1, J_2, \dots, J_n .

- Let $J_{i_1}, J_{i_2}, \dots, J_{i_n}$ be an optimum schedule with fewest inversions.
- If schedule has no inversions then it is identical to **SJF** schedule and we are done.
- Otherwise there is an $1 \leq \ell < n$ such that $i_\ell > i_{\ell+1}$ since schedule has inversion among two adjacently scheduled jobs

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- Otherwise there is an $1 \leq \ell < n$ such that $i_\ell > i_{\ell+1}$ since schedule has inversion among two adjacently scheduled jobs

Claim

The schedule obtained from $J_{i_1}, J_{i_2}, \dots, J_{i_n}$ by exchanging/swapping positions of jobs J_{i_ℓ} and $J_{i_{\ell+1}}$ is also optimal and has one fewer inversion.

Assuming claim we obtain a contradiction and hence optimum schedule with fewest inversions must be the **SJF** schedule.

A Weighted Version

- n jobs J_1, J_2, \dots, J_n . J_i has non-negative processing time p_i and a non-negative weight w_i
- One server/machine/person available to process jobs.
- Schedule/order the jobs to minimize total or average *waiting time*
- Waiting time of J_i in schedule σ : sum of processing times of all jobs scheduled before J_i
- Goal: minimize total *weighted* waiting time.

	J_1	J_2	J_3	J_4	J_5	J_6
<i>time</i>	3	4	1	8	2	6
<i>weight</i>	10	5	2	100	1	1

$$0 + 3 \times 10 + (3+4) \times 5 + (3+4+1) \times 2 + \dots + \dots + \dots$$

A Weighted Version

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$$P_1 \text{ first} : T_1 = P_1 w_1 + (P_1 + P_2) w_2$$

$$P_2 \text{ first} : T_2 = P_2 w_2 + (P_1 + P_2) w_1$$

$$T_1 < T_2$$

$$\cancel{P_1 w_1} + P_1 w_2 + \cancel{P_2 w_2} < \cancel{P_2 w_2} + \cancel{P_1 w_1} + P_2 w_1$$

$$P_1 w_2 < P_2 w_1$$

$$\frac{P_1}{w_1} < \frac{P_2}{w_2}$$

Part III


Scheduling to Minimize Lateness

Scheduling to Minimize Lateness

- ① Given jobs J_1, J_2, \dots, J_n with deadlines and processing times to be scheduled on a single resource.
- ② If a job i starts at time s_i then it will finish at time $f_i = s_i + t_i$, where t_i is its processing time. d_i : deadline.
- ③ The lateness of a job is $\ell_i = \max(0, f_i - d_i)$.
- ④ Schedule all jobs such that $L = \max \ell_i$ is minimized.

Scheduling to Minimize Lateness

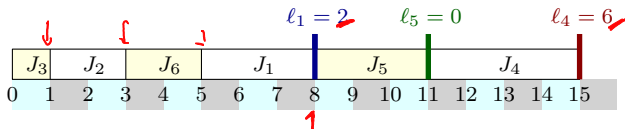
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	J_1	J_2	J_3	J_4	J_5	J_6
 t_i	3	2	1	4	3	2
d_i	6	8	9	9	14	15

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t_i	3	2	1	4	3	2
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Greedy Template

Initially R is the set of all requests

→ $curr_time = 0$

→ $max_lateness = 0$

→ while R is not empty do

→ choose $i \in R$

→ $curr_time = curr_time + t_i$

→ if ($curr_time > d_i$) then

→ $max_lateness = \max(curr_time - d_i, max_lateness)$

return $max_lateness$

Greedy Template

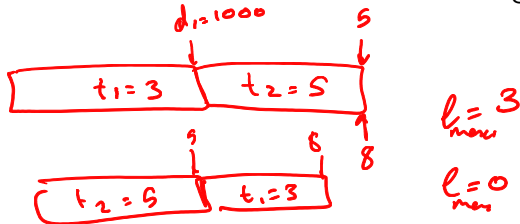
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while  $R$  is not empty do  
   $\rightarrow$  choose  $i \in R$   
     $curr\_time = curr\_time + t_i$   
    if ( $curr\_time > d_i$ ) then  
       $max\_lateness = \max(curr\_time - d_i, max\_lateness)$   
  
return  $max\_lateness$ 
```

Main task: Decide the order in which to process jobs in R

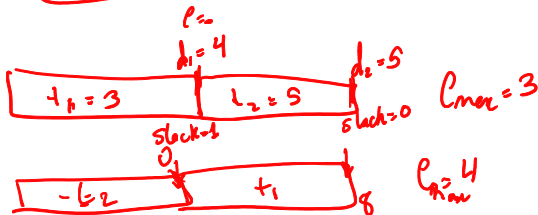
Three Algorithms

- ➔ ① Shortest job first — sort according to t_i .
- ➔ ② Shortest slack first — sort according to $\underline{d_i} - \underline{t_i}$.
- ➔ ③ EDF = Earliest deadline first — sort according to d_i .

SJF:



SSF:



Three Algorithms

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Counter examples for first two: exercise

Earliest Deadline First

Theorem

Greedy with EDF rule minimizes maximum lateness.

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Proof via an exchange argument.

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Idle time: time during which machine is not working.

Earliest Deadline First

Theorem

Greedy with EDF rule minimizes maximum lateness.

Proof via an exchange argument.

Idle time: time during which machine is not working.

Lemma

If there is a feasible schedule then there is one with no idle time before all jobs are finished.

Inversions

EDF = Earliest Deadline First

Assume jobs are sorted such that $d_1 \leq d_2 \leq \dots \leq d_n$. Hence **EDF** schedules them in this order.

Definition

A schedule S is said to have an **inversion** if there are jobs i and j such that S schedules i before j , but $d_i > d_j$.

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Assume jobs are sorted such that $d_1 \leq d_2 \leq \dots \leq d_n$. Hence **EDF** schedules them in this order.

Definition

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Claim

If a schedule S has an inversion then there is an inversion between two adjacently scheduled jobs.

Proof: exercise.

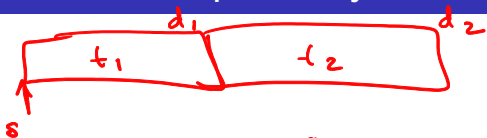
Proof sketch of Optimality of EDF

- Let S be an optimum schedule with smallest number of inversions.
- If S has no inversions then this is same as **EDF** and we are done.
- Else S has two adjacent jobs i and j with $d_i > d_j$.
- Swap positions of i and j to obtain a new schedule S'

Claim

Maximum lateness of S' is no more than that of S . And S' has strictly fewer inversions than S .

Proof sketch of Optimality of EDF



$$\rightarrow J_1 \text{ first} : \max \left\{ 0, \overbrace{s+t_1-d_1}^{a_1}, \overbrace{s+t_1+t_2-d_2}^{b_1} \right\}$$
$$\rightarrow J_2 \text{ first} : \max \left\{ 0, \overbrace{s+t_2-d_2}^{a_2}, \overbrace{s+t_1+t_2-d_1}^{b_2} \right\}$$

$$d_1 < d_2$$
$$-d_1 > -d_2 \Rightarrow b_2 > b_1 > a_2$$
$$b_2 > a_1$$

Part IV

Maximum Weight Subset of Elements: Cardinality and Beyond

Picking k elements to maximize total weight

- 1 Given n items each with non-negative weights/profits and integer $1 \leq k \leq n$.
- 2 Goal: pick k elements to **maximize** total weight of items picked.

	e_1	e_2	e_3	e_4	e_5	e_6
<i>weight</i>	3	2	1	4	3	2

$k = 2$:

$k = 3$:

$k = 4$:

Greedy Template

```
N is the set of all elements  $X \leftarrow \emptyset$ 
(*  $X$  will store all the elements that will be picked *)
while  $|X| < k$  and N is not empty do
    choose  $e_j \in N$  of maximum weight
    add  $e_j$  to  $X$ 
    remove  $e_j$  from N
return the set  $X$ 
```

Remark: One can rephrase algorithm simply as sorting elements in decreasing weight order and picking the top k elements but the above template generalizes to other settings a bit more easily.

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Remark: One can rephrase algorithm simply as sorting elements in decreasing weight order and picking the top k elements but the above template generalizes to other settings a bit more easily.

Theorem

Greedy is optimal for picking k elements of maximum weight.

A more interesting problem

- 1 Given n items $N = \{e_1, e_2, \dots, e_n\}$. Each item e_i has a non-negative weight w_i .
- 2 Items partitioned into h sets N_1, N_2, \dots, N_h . Think of each item having one of h colors.
- 3 Given integers k_1, k_2, \dots, k_h and another integer k
- 4 Goal: pick k elements such that no more than k_i from N_i to **maximize** total weight of items picked.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
<i>weight</i>	9	5	4	7	5	2	1

$$N_1 = \{e_1, e_2, e_3\}, N_2 = \{e_4, e_5\}, N_3 = \{e_6, e_7\}$$

$$k = 4, \underbrace{k_1 = 2}, \underbrace{k_2 = 1}, \underbrace{k_3 = 2}$$

Greedy Template

```
N is the set of all elements X  $\leftarrow \emptyset$ 
(* X will store all the elements that will be picked *)
while N is not empty do
    N' = { $e_i \in N \mid X \cup \{e_i\}$  is feasible}
    if N' =  $\emptyset$  then break
    choose  $e_j \in N'$  of maximum weight
    add  $e_j$  to X
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Greedy Template

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N is the set of all elements  $X \leftarrow \emptyset$ 
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return the set X
```

Theorem

Greedy is optimal for the problem on previous slide.

Proof: exercise after class.

Special case of general phenomenon of Greedy working for maximum weight independent set in a **matroid**. Beyond scope of course.

Part V

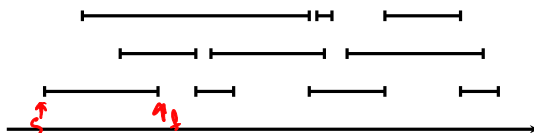
Interval Scheduling

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Problem (Interval Scheduling)

Input: *A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms).*

Goal: *Schedule as many jobs as possible*



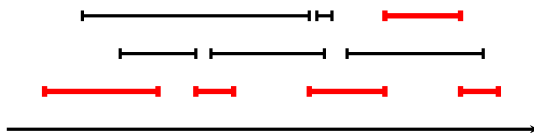
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Problem (Interval Scheduling)

Input: *A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms).*

Goal: *Schedule as many jobs as possible*

- ① *Two jobs with overlapping intervals cannot both be scheduled!*



Greedy Template

```
R is the set of all requests  
X  $\leftarrow \emptyset$  (* X will store all the jobs that will be scheduled *)  
while R is not empty do  
    choose  $i \in R$   
    add  $i$  to X  
    remove from R all requests that overlap with  $i$   
return the set X
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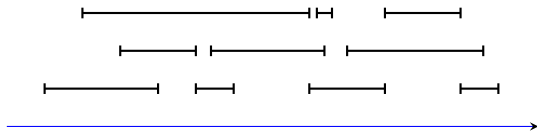

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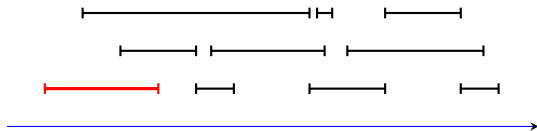
Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.



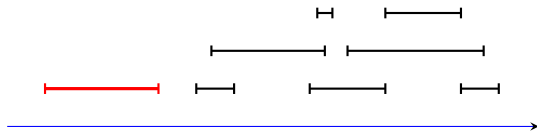
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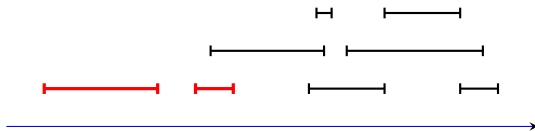
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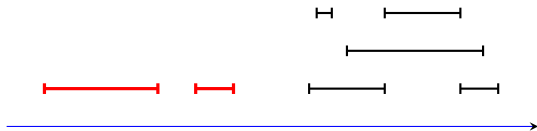
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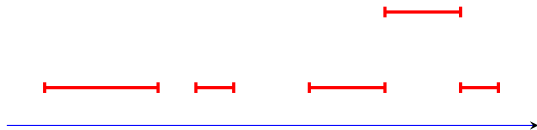
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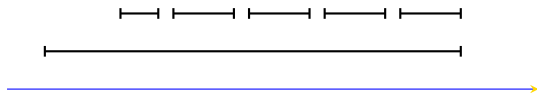


Figure: Counter example for earliest start time

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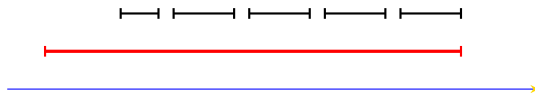


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Figure: Counter example for earliest start time

Smallest Processing Time

Process jobs in the order of processing time, starting with jobs that require the shortest processing.



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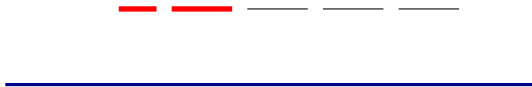
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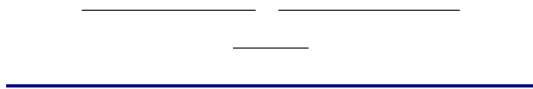


Figure: Counter example for smallest processing time

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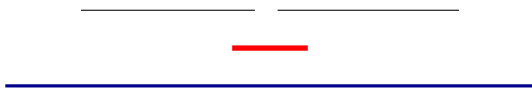


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Process jobs in that have the fewest “conflicts” first.



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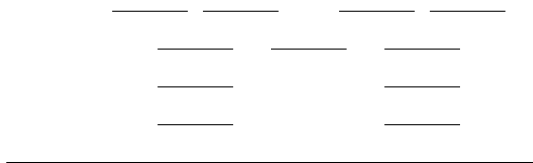


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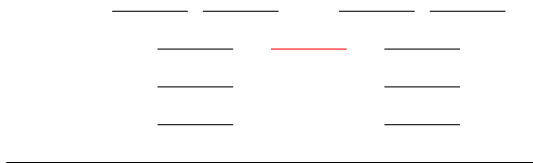


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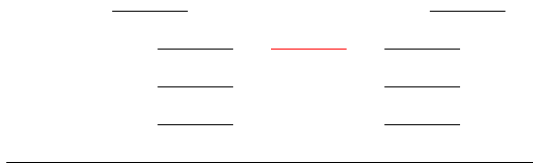


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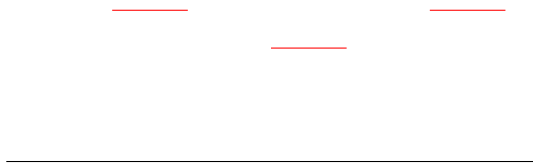
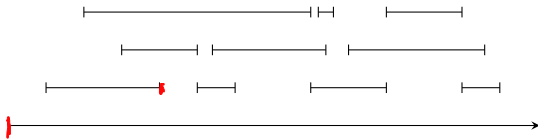


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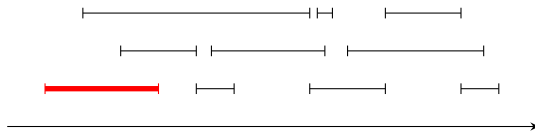
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Process jobs in the order of their finishing times, beginning with those that finish earliest.



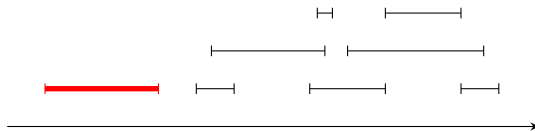
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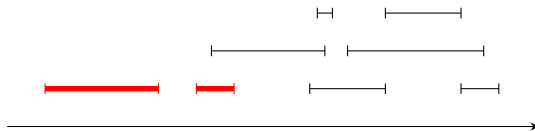
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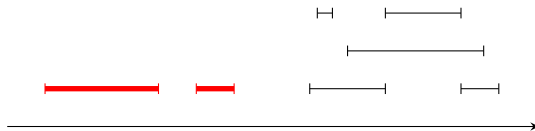
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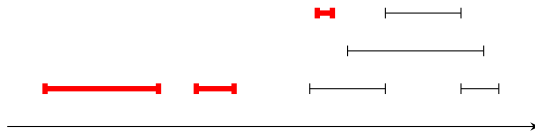
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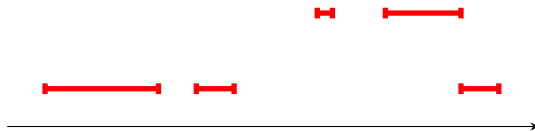
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Optimal Greedy Algorithm

```
R is the set of all requests
X  $\leftarrow \emptyset$  (* X stores the jobs that will be scheduled *)
while R is not empty
   $\rightarrow$  choose  $i \in R$  such that finishing time of  $i$  is smallest
  add  $i$  to X
  remove from R all requests that overlap with  $i$ 
return X
```

Theorem

The greedy algorithm that picks jobs in the order of their finishing times is optimal.

Proving Optimality

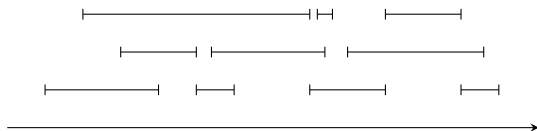
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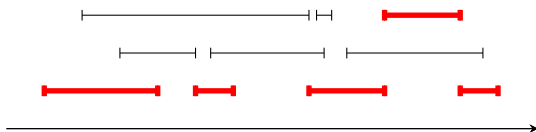
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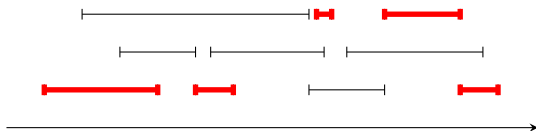
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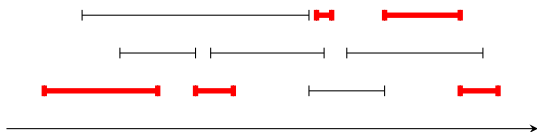
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- 2 For a set of requests R , let O be an optimal set and let X be the set returned by the greedy algorithm. Then $O = X$? Not likely!



Instead we will show that $|O| = |X|$

Proof of Optimality: Key Lemma

Lemma

Let i_1 be first interval picked by Greedy. There exists an optimum solution that contains i_1 .

Proof.

Let O be an arbitrary optimum solution. If $i_1 \in O$ we are done.

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→ **Claim:** If $i_1 \notin O$ then there is exactly one interval $j_1 \in O$ that conflicts with i_1 . (proof later)

- 1 Form a new set O' by removing j_1 from O and adding i_1 , that is $O' = (O - \{j_1\}) \cup \{i_1\}$.
- 2 From claim, O' is a *feasible* solution (no conflicts).
- 3 Since $|O'| = |O|$, O' is also an optimum solution and it contains i_1 . □

Proof of Claim

Claim

If $i_1 \notin O$, there is exactly one interval $j_1 \in O$ that conflicts with i_1 .

Proof.

- 1 If no $j \in O$ conflicts with i_1 then O is not optimal!
- 2 Suppose $j_1, j_2 \in O$ such that $j_1 \neq j_2$ and both j_1 and j_2 conflict with i_1 .
- 3 Since i_1 has earliest finish time, j_1 and i_1 overlap at $f(i_1)$.
- 4 For same reason j_2 also overlaps with i_1 at $f(i_1)$.
- 5 Implies that j_1, j_2 overlap at $f(i_1)$ but intervals in O cannot overlap.

See figure in next slide.



Figure for proof of Claim

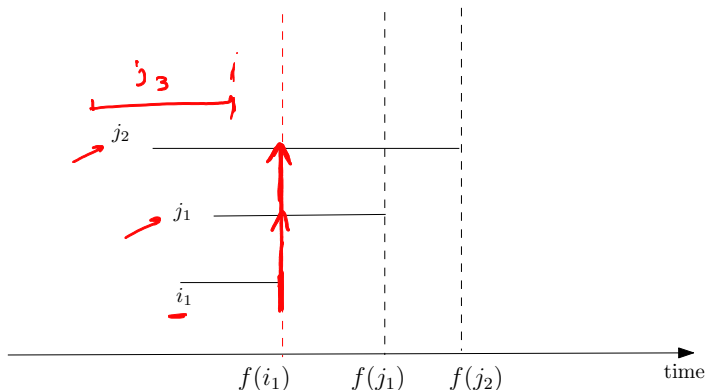


Figure: Since i_1 has the earliest finish time, any interval that conflicts with it does so at $f(i_1)$. This implies j_1 and j_2 conflict.

Proof of Optimality of Earliest Finish Time First

Proof by Induction on number of intervals.

Base Case: $n = 1$. Trivial since Greedy picks one interval.

Induction Step: Assume theorem holds for $i < n$.

Let I be an instance with n intervals

I' : I with i_1 and all intervals that overlap with i_1 removed

$G(I)$, $G(I')$: Solution produced by Greedy on I and I'

From Lemma, there is an optimum solution O to I and $i_1 \in O$.

Let $O' = O - \{i_1\}$. O' is a solution to I' .

$$\begin{aligned} |G(I)| &= 1 + |G(I')| \quad (\text{from Greedy description}) \\ &\geq 1 + |O'| \quad (\text{By induction, } G(I') \text{ is optimum for } I') \\ &= |O| \end{aligned}$$



Implementation and Running Time

```
Initially  $R$  is the set of all requests  
 $X \leftarrow \emptyset$  (*  $X$  stores the jobs that will be scheduled *)  
while  $R$  is not empty  
    choose  $i \in R$  such that finishing time of  $i$  is least  
    if  $i$  does not overlap with requests in  $X$   
        add  $i$  to  $X$   
    remove  $i$  from  $R$   
return the set  $X$ 
```

- Presort all requests based on finishing time. $O(n \log n)$ time
- Now choosing least finishing time is $O(1)$
- Keep track of the finishing time of the last request added to A . Then check if starting time of i later than that
- Thus, checking non-overlapping is $O(1)$
- Total time $O(n \log n + n) = O(n \log n)$

Comments

- Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.
- All requests need not be known at the beginning. Such *online* algorithms are a subject of research

Weighted Interval Scheduling

Suppose we are given n jobs. Each job i has a start time s_i , a finish time f_i , and a weight w_i . We would like to find a set S of compatible jobs whose total weight is maximized. Which of the following greedy algorithms finds the optimum schedule?

- (A) Earliest start time first.
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Weighted problem can be solved via dynamic programming. See notes.

Greedy Analysis: Overview

- ① **Greedy's first step leads to an optimum solution.** Show that there is an optimum solution leading from the first step of Greedy and then use induction. Example, Interval Scheduling.
- ② **Greedy algorithm stays ahead.** Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.
- ③ **Structural property of solution.** Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning (see Kleinberg-Tardos book).
- ④ **Exchange argument.** Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality. Example, Minimizing lateness.

Takeaway Points

- ① Greedy algorithms come naturally but often are incorrect. A proof of correctness is an absolute necessity.
- ② *Exchange* arguments are often the key proof ingredient. Focus on why the first step of the algorithm is correct: need to show that there is an optimum/correct solution with the first step of the algorithm.
- ③ Thinking about correctness is also a good way to figure out which of the many greedy strategies is likely to work.