CS/ECE 374 A (Spring 2020)
Homework 7 (due Mar 26 Thursday at 10am)

Instructions: As in previous homeworks.

Problem 7.1:

(a) (8.5 points) We are given a DFA $M = (Q, \Sigma, \delta, s, A)$ over the alphabet $\Sigma = \{0, 1\}$ with $m = |Q|$ states, and we are given a string $x = a_1 \cdots a_n$ of length n ($a_i \in \{0, 1\}$). We want to find a string $y = b_1 \cdots b_n$ of length n that is accepted by M and is "closest" to x, in the sense of minimizing the distance $d(x, y) = |\{i : a_i \neq b_i\}|$ (i.e., the number of differing bits).

Describe an efficient dynamic programming algorithm[1] to solve this problem. The algorithm should output not only the minimum distance but also the closest string y. Analyze the running time as a function of n and m.

(b) (1.5 points) Describe how to modify your algorithm and analysis if the given automaton M is an NFA instead. You may assume that the given NFA does not have ε-transitions (since there are efficient algorithms to remove ε-transitions without increasing the number of states).

(Note: if the analysis is done carefully, the running time in (a) should be better than in (b).)

(Note: the analogous problem for regular expressions can similarly be solved, since regular expressions can be efficiently converted to NFAs.)

Problem 7.2: Given an unordered binary tree T, a *preorder traversal* is a list (an ordering) of the nodes of T that can be obtained recursively by the following rules:

- If T has a single node r, then the list $\langle r \rangle$ is a preorder traversal.
- If T has root r and has subtrees T_1 and T_2 at r's two children, and L_1 and L_2 are valid preorder traversals of T_1 and T_2 respectively, then $\langle r \rangle \cdot L_1 \cdot L_2$ and $\langle r \rangle \cdot L_2 \cdot L_1$ are both preorder traversals of T. Here, $\cdot$ denotes concatenation. (You may assume that all non-leaf nodes have degree 2.)

Let $d(\cdot, \cdot)$ be a given distance function, which can be evaluated in constant time.

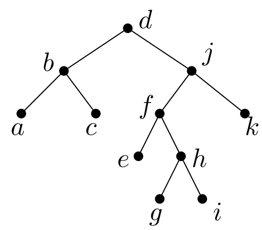
(a) (8.0 points) Given an unordered binary tree T with n nodes, we want to find a preorder traversal with the minimum cost. Here, the *cost* of $\langle v_1, v_2, \ldots, v_n \rangle$ is defined to be $d(v_1, v_2) + d(v_2, v_3) + \cdots + d(v_{n-1}, v_n)$.

Describe an efficient dynamic programming algorithm to compute the cost of an optimal traversal. Analyze its worst-case running time. (Note: a correct solution with $O(n^2)$ running time gets full credit; $O(n^3)$ gets a maximum of 6.0 points.)

[1] See the general note from HW6 on what we expect in a dynamic programming solution.

(b) (2.0 points) Modify your algorithm and/or analysis to obtain a better running time in the special case when T is a *balanced* binary tree with $O(\log n)$ height.

For example: in the following tree, $\langle d, j, f, e, h, g, i, k, b, a, c\rangle$ and $\langle d, b, c, a, j, k, e, f, h, i, g\rangle$ are two preorder traversals (and there are many more).



Problem 7.3: The motivation behind this problem is how to divide a set of data points into a given number k of clusters.

Given a set P of n points in 2D, a *binary space partition (BSP)* is a binary tree where each node v stores a subset of points $P(v) \subseteq P$, and for every non-leaf node v with children v_1 and v_2, we have one of the following:

- $P(v_1) = \{p \in P(v) \mid p.x \leq m\}$ and $P(v_2) = \{p \in P(v) \mid p.x > m\}$ for some value m; or
- $P(v_1) = \{p \in P(v) \mid p.y \leq m\}$ and $P(v_2) = \{p \in P(v) \mid p.y > m\}$ for some value m.

In other words, $P(v)$ is split into two subsets $P(v_1)$ and $P(v_2)$ by cutting with either a vertical line $x = m$ or a horizontal line $y = m$. (Here, $p.x$ and $p.y$ denote the x- and y-coordinate of a point p respectively.) At the root r, we have $P(r) = P$.

For a set Q of points, define $c(Q) = (\max_{q\in Q} q.x - \min_{q\in Q} q.x) \cdot (\max_{q\in Q} q.y - \min_{q\in Q} q.y)$ (i.e., it is the area of the smallest axis-aligned rectangle containing Q).

Given a set P of n points in 2D and an integer k, we want to find a BSP with k leaves to minimize the cost function $\sum_{\text{leaf } v} c(P(v))$.

Describe (and analyze) an efficient dynamic programming algorithm to compute the cost of an optimal BSP for this problem.

An example of a (not necessarily optimal) BSP with $k = 8$ leaves is given below: