Describe a DFA that accepts the following language over the alphabet $\Sigma = \{0, 1\}$.

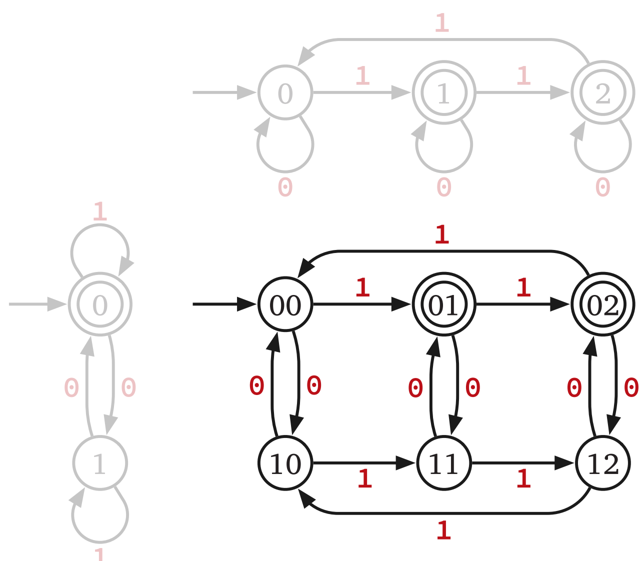
**1** DFA for all strings in which the number of 0s is even and the number of 1s is *not* divisible by 3.

**Solution:**

We use a standard product construction of two DFAs, one accepting strings with an even number of 0s, and the other accepting strings where the number of 1s is not a multiple of 3.
The product DFA has six states, each labeled with a pair of integers, one indicating the number 0s read modulo 2, the other indicating the number of 1s read modulo 3.

$$Q := \{0,1\} \times \{0,1,2\}$$
$$s := (0,0)$$
$$A := \{(0,1),(0,2)\}$$
$$\delta((q,r),0) := (q + 1 \bmod 2,\ r)$$
$$\delta((q,r),1) := (q,\ r + 1 \bmod 3)$$

In this case, the product DFA is simple enough that we can just draw it out in full. I have drawn the two factor DFAs (in gray) to the left and above for reference.




**2** DFA for all strings that are **both** the binary representation of an integer divisible by 3 **and** the ternary (base-3) representation of an integer divisible by 4.

For example, the string 1100 is an element of this language, because it represents $2^3 + 2^2 = 12$ in binary and $3^3 + 3^2 = 36$ in ternary.

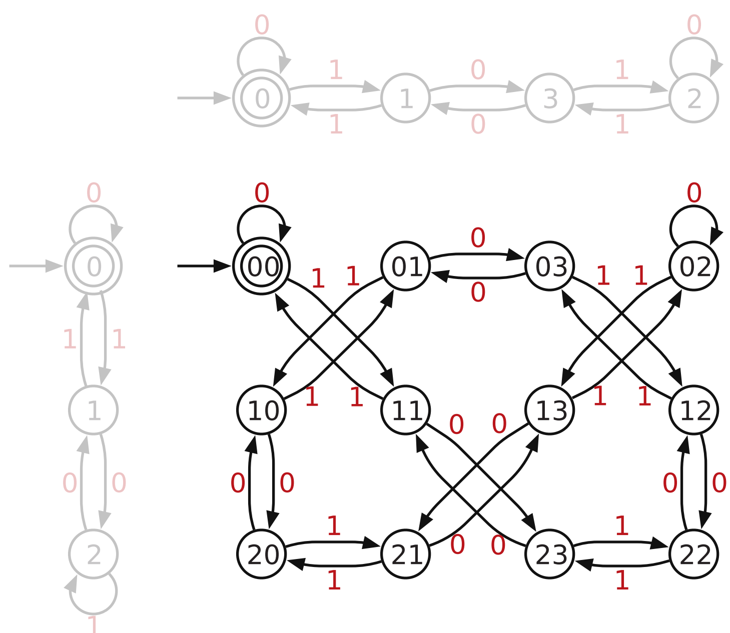
**Solution:**

Again, we use a standard product construction of two DFAs, one accepting binary strings divisible by 3, the other accepting ternary strings divisible by 4. The product DFA has twelve states, each labeled with a pair of integers: The binary value read so far modulo 3, and the ternary value read so far modulo 4.

$$
\begin{aligned}
Q &:= \{0,1,2\} \times \{0,1,2,3\} \\
s &:= (0,0) \\
A &:= \{(0,0)\}
\end{aligned}
$$

$$
\begin{aligned}
\delta((q,r),\mathtt{0}) &:= (2q \bmod 3, \quad\quad 3r \bmod 4) \\
\delta((q,r),\mathtt{1}) &:= (2q+1 \bmod 3,\ \ 3r+1 \bmod 4)
\end{aligned}
$$

For reference, here is a drawing of the DFA, with the two factor DFAs (in gray) to the left and above; we would not expect you to draw this, especially on exams. More importantly we would expect you ***not*** to draw this, ***especially*** on exams. The states of the factor DFA that maintains ternary-value-mod-4 are deliberately "out of order" to simplify the drawing.




**3** DFA for all strings $w$ such that $\binom{|w|}{2} \bmod 6 = 4$. (**Hint:** Maintain both $\binom{|w|}{2} \bmod 6$ and $|w| \bmod 6$.)

## Solution:

Our DFA has 36 states, each labeled with a pair of integers representing $\binom{|x|}{2} \bmod 6$ and $|x| \bmod 6$, where $x$ is the prefix of the input read so far.

$$
\begin{aligned}
Q &:= \{0,1,2,3,4,5\} \times \{0,1,2,3,4,5\} \\
s &:= \{(0,0)\} \\
A &:= \{(4,r) \mid r \in \{0,1,2,3,4,5\}\}
\end{aligned}
$$

$$
\begin{aligned}
\delta((q,r),\mathtt{0}) &:= (q+r \bmod 6,\ r+1 \bmod 6) \\
\delta((q,r),\mathtt{1}) &:= (q+r \bmod 6,\ r+1 \bmod 6)
\end{aligned}
$$

The transition function exploits the identity $\binom{n+1}{2} = \binom{n}{2} + n$.

## Solution:

The language is identical to the set of strings $w$ such that $|w| \bmod 12 \in \{4, 7\}$. This language can be accepted using a 12-state DFA.

**4** (**Hard.**) All strings $w$ such that $F_{\#(10,w)} \bmod 10 = 4$, where $\#(10, w)$ denotes the number of times 10 appears as a substring of $w$, and $F_n$ is the $n$th Fibonacci number:

$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{otherwise} \end{cases}$$

## Solution:

Our DFA has 200 states, each labeled with three values:

- $F_k \bmod 10$, where $k$ is the number of times we have seen the substring 10.
- $F_{k+1} \bmod 10$, where $k$ is the number of times we have seen the substring 10.
- The last symbol read (or 0 if we have read nothing yet)

Here is the formal description:

$$\begin{aligned} Q &:= \{0,1,2,3,4,5,6,7,8,9\} \times \{0,1,2,3,4,5,6,7,8,9\} \times \{0,1\} \\ s &:= \{(0,1,0)\} \\ A &:= \{(4,r,a) \mid r \in \{0,1,2,3,4,5,6,7,8,9\} \text{ and } a \in \{0,1\}\} \end{aligned}$$

$$\begin{aligned} \delta((q,r,0),0) &:= (q,r,0) \\ \delta((q,r,1),0) &:= (r, q+r \bmod 10, 0) \\ \delta((q,r,0),1) &:= (q,r,1) \\ \delta((q,r,1),1) &:= (q,r,1) \end{aligned}$$

The transition function exploits the recursive definition $F_{k+1} = F_k + F_{k-1}$.

## Solution:

The Fibonacci numbers modulo 10 define a repeating sequence with period 60. So this language can be accepted by a DFA with "only" 120 states.