Prove that the following languages are undecidable.

See outline of how to solve such problems in the original problem set.

1 AcceptIllini := $\{\langle M \rangle \mid M$ accepts the string $ILLINI\}$

Solution:

For the sake of argument, suppose there is an algorithm DecideAcceptIllini that correctly decides the language AcceptIllini. Then we can solve the halting problem as follows:

```
DecideHalt(⟨M, w⟩):
  Encode the following Turing machine M':
    M'(x):
      run M on input w
      return True
  if DecideAcceptIllini(⟨M'⟩)
    return True
  else
    return False
```

We prove this reduction correct as follows:

$\Longrightarrow$ Suppose M halts on input w.
Then M' accepts *every* input string x.
In particular, M' accepts the string $ILLINI$.
So **DecideAcceptIllini** accepts the encoding $\langle M' \rangle$.
So **DecideHalt** correctly accepts the encoding $\langle M, w \rangle$.

$\Longleftarrow$ Suppose M does not halt on input w.
Then M' diverges on *every* input string x.
In particular, M' does not accept the string $ILLINI$.
So **DecideAcceptIllini** rejects the encoding $\langle M' \rangle$.
So **DecideHalt** correctly rejects the encoding $\langle M, w \rangle$.

In both cases, **DecideHalt** is correct. But that's impossible, because **Halt** is undecidable. We conclude that the algorithm **DecideAcceptIllini** does not exist.

As usual for undecidablility proofs, this proof invokes *four* distinct Turing machines:

- The hypothetical algorithm **DecideAcceptIllini**.
- The new algorithm **DecideHalt** that we construct in the solution.
- The arbitrary machine M whose encoding is part of the input to **DecideHalt**.
- The special machine M' whose encoding **DecideHalt** constructs (from the encoding of M and w) and then passes to **DecideAcceptIllini**.

2 AcceptThree := $\{\langle M \rangle \mid M$ accepts exactly three strings$\}$

Solution:

For the sake of argument, suppose there is an algorithm **DecideAcceptThree** that correctly decides the language AcceptThree. Then we can solve the halting problem as follows:

```
DecideHalt(⟨M, w⟩):
  Encode the following Turing machine M′:
    M′(x):
      run M on input w
      if x = ε or x = 0 or x = 1
          return True
      else
          return False
  if DecideAcceptThree(⟨M′⟩)
      return True
  else
      return False
```

We prove this reduction correct as follows:

$\Longrightarrow$ Suppose M halts on input w.
Then M' accepts exactly three strings: ε, 0, and 1.
So **DecideAcceptThree** accepts the encoding $\langle M' \rangle$.
So **DecideHalt** correctly accepts the encoding $\langle M, w \rangle$.

$\Longleftarrow$ Suppose M does not halt on input w.
Then M' diverges on *every* input string x.
In particular, M' does not accept exactly three strings (because $0 \neq 3$).
So **DecideAcceptThree** rejects the encoding $\langle M' \rangle$.
So **DecideHalt** correctly rejects the encoding $\langle M, w \rangle$.

In both cases, **DecideHalt** is correct. But that's impossible, because Halt is undecidable. We conclude that the algorithm **DecideAcceptThree** does not exist.

3 AcceptPalindrome $:= \{\langle M \rangle \mid M \text{ accepts at least one palindrome}\}$

Solution:

For the sake of argument, suppose there is an algorithm **DecideAcceptPalindrome** that correctly decides the language **AcceptPalindrome**. Then we can solve the halting problem as follows:

```
DecideHalt(⟨M, w⟩):
  Encode the following Turing machine M′:
    M′(x):
      run M on input w
      return True
  if DecideAcceptPalindrome(⟨M′⟩)
      return True
  else
      return False
```

We prove this reduction correct as follows:

$\Longrightarrow$ Suppose M halts on input w.
Then M' accepts *every* input string x.
In particular, M' accepts the palindrome $RACECAR$.
So **DecideAcceptPalindrome** accepts the encoding $\langle M' \rangle$.
So **DecideHalt** correctly accepts the encoding $\langle M, w \rangle$.

$\Longleftarrow$ Suppose M does not halt on input w.
Then M' diverges on *every* input string x.
In particular, M' does not accept any palindromes.
So **DecideAcceptPalindrome** rejects the encoding $\langle M' \rangle$.
So **DecideHalt** correctly rejects the encoding $\langle M, w \rangle$.

In both cases, **DecideHalt** is correct. But that's impossible, because HALT is undecidable. We conclude that the algorithm **DecideAcceptPalindrome** does not exist.

Yes, this is ***exactly*** the same proof as for problem 1.