Proving that a problem X is NP-hard requires several steps:

- Choose a problem Y that you already know is NP-hard (because we told you so in class).
- Describe an algorithm to solve Y, using an algorithm for X as a subroutine. Typically this algorithm has the following form: Given an instance of Y, transform it into an instance of X, and then call the magic black-box algorithm for X.
- ***Prove*** that your algorithm is correct. This always requires two separate steps, which are usually of the following form:
  - ***Prove*** that your algorithm transforms "good" instances of Y into "good" instances of X.
  - ***Prove*** that your algorithm transforms "bad" instances of Y into "bad" instances of X. Equivalently: Prove that if your transformation produces a "good" instance of X, then it was given a "good" instance of Y.
- Argue that your algorithm for Y runs in polynomial time.

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1 A ***Hamiltonian cycle*** in a graph G is a cycle that goes through every vertex of G exactly once. Deciding whether an arbitrary graph contains a Hamiltonian cycle is NP-hard.

A ***tonian cycle*** in a graph G is a cycle that goes through at least *half* of the vertices of G. Prove that deciding whether a graph contains a tonian cycle is NP-hard.

2 *Big Clique* is the following decision problem: given a graph $G = (V, E)$, does G have a clique of size at least $n/2$ where $n = |V|$ is the number of nodes? Prove that *Big Clique* is NP-hard.

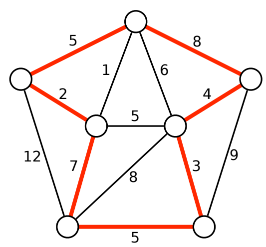
3 Recall the following kCOLOR problem: Given an undirected graph G, can its vertices be colored with k colors, so that every edge touches vertices with two different colors?

3.A. Describe a direct polynomial-time reduction from 3COLOR to 4COLOR.

3.B. Prove that kCOLOR problem is NP-hard for any $k \geq 3$.

To think about later:

4 Let G be an undirected graph with weighted edges. A Hamiltonian cycle in G is ***heavy*** if the total weight of edges in the cycle is at least half of the total weight of all edges in G. Prove that deciding whether a graph contains a heavy Hamiltonian cycle is NP-hard.



A heavy Hamiltonian cycle. The cycle has total weight 34; the graph has total weight 67.