Proving Non-regularity

Lecture 6

Friday, February 7, 2020

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- Each DFA M can be represented as a string over a finite alphabet Σ by appropriate encoding
- Hence number of regular languages is *countably infinite*
- Number of languages is *uncountably infinite*
- Hence there must be a non-regular language!

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$M = (Q, \Sigma, \delta, s, A)$: DFA *for language* $L \subseteq \Sigma^*$.
If $\delta^*(s, xw) \in A$ *and* $\delta^*(s, yw) \notin A$ *then* $\delta^*(s, x) \neq \delta^*(s, y)$.

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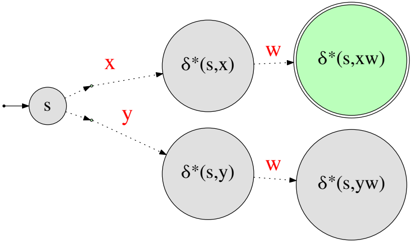
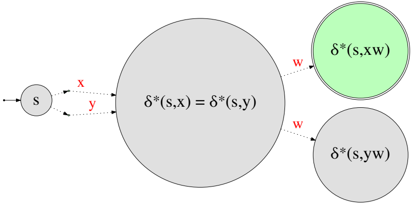
$\implies A \ni \delta^*(s, xw) \notin A$. Impossible! □

Proof by figures

Possible	Not possible
	

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How do we formalize intuition and come up with a formal proof?

Proof by Contradiction

- Suppose L is regular. Then there is a DFA M such that $L(M) = L$.
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What states does M reach on the above strings? Let $q_i = \delta^*(s, 0^i)$.

By pigeon hole principle $q_i = q_j$ for some $0 \le i < j \le n$.
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M should accept 0^i1^i but then it will also accept 0^j1^i where $i \neq j$. This contradicts the fact that M accepts L. Thus, there is no DFA for L.

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Example: 000 and 0000 are indistinguishable with respect to the language $L = \{w \mid w \text{ has } 00 \text{ as a substring}\}$

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Lemma

Suppose $L = L(M)$ *for some* DFA $M = (Q, \Sigma, \delta, s, A)$ *and suppose* x, y *are distinguishable with respect to* L. *Then* $\delta^*(s, x) \neq \delta^*(s, y)$.

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Proof.

Since x, y are distinguishable let w be the distinguishing suffix. If $\delta^*(s, x) = \delta^*(s, y)$ then M will either accept both the strings xw, yw, or reject both. But exactly one of them is in L, a contradiction. □

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Suppose there is a DFA $M = (Q, \Sigma, \delta, s, A)$ that accepts L. Let $|Q| = n$.
If $n < |F|$ then by pigeon hole principle there are two strings $x, y \in F$, $x \neq y$ such that $\delta^*(s, x) = \delta^*(s, y)$ but x, y are distinguishable.

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Implies that there is w such that exactly one of xw, yw is in L.
However, M's behavior on xw and yw is exactly the same and hence M will accept both xw, yw or reject both. A contradiction. □

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Corollary

If L has an infinite fooling set F then L is not regular.

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Proof.

Suppose for contradiction that $L = L(M)$ for some DFA M with n states.
Any subset F' of F is a fooling set. (Why?) Pick $F' \subseteq F$ arbitrarily such that $|F'| > n$. By preceding theorem, we obtain a contradiction. □

Examples

- $\{0^k 1^k \mid k \geq 0\}$

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Why?

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- Suppose $a_1 a_2 \ldots a_k$ and $b_1 b_2 \ldots b_k$ are two distinct bitstrings of length k
- Let i be first index where $a_i \neq b_i$
- $y = 0^{k-i-1}$ is a distinguishing suffix for the two strings

How do pick a fooling set

How do we pick a fooling set F?

- If x, y are in F and $x \neq y$ they should be distinguishable! Of course.
- All strings in F except maybe one should be prefixes of strings in the language L.
  For example if $L = \{0^k 1^k \mid k \geq 0\}$ do not pick 1 and 10 (say). Why?

Part I

Non-regularity via closure properties

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$L = \{$bitstrings with equal number of 0s and 1s$\}$

$L' = \{0^k 1^k \mid k \geq 0\}$

Suppose we have already shown that L' is non-regular. Can we show that L is non-regular without using the fooling set argument from scratch?

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Claim: The above and the fact that L' is non-regular implies L is non-regular. Why?

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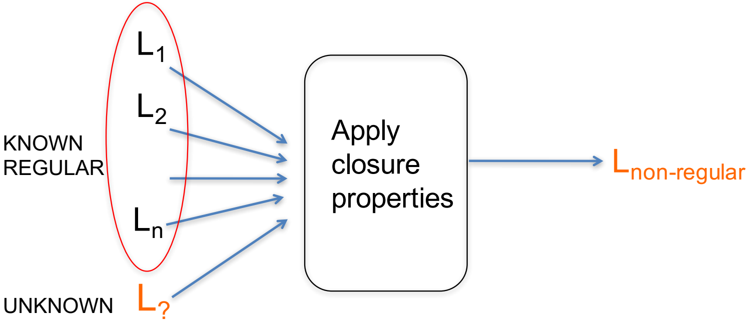
Suppose L is regular. Then since $L(0^*1^*)$ is regular, and regular languages are closed under intersection, L' also would be regular. But we know L' is not regular, a contradiction.

Non-regularity via closure properties

General recipe:



Proving non-regularity: Summary

- Method of distinguishing suffixes. To prove that L is non-regular find an infinite fooling set.
- Closure properties. Use existing non-regular languages and regular languages to prove that some new language is non-regular.
- Pumping lemma. We did not cover it but it is sometimes an easier proof technique to apply, but not as general as the fooling set technique.

Part II

Myhill-Nerode Theorem

Indistinguishability

Recall:

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Given language L over Σ define a relation $\equiv_L$ over strings in Σ^* as follows: $x \equiv_L y$ iff x and y are indistinguishable with respect to L.

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Claim

Let x, y be two distinct strings. If x, y belong to the same equivalence class of $\equiv_L$ then x, y are indistinguishable. Otherwise they are distinguishable.

Corollary

If $\equiv_L$ is finite with n equivalence classes then there is a fooling set F of size n for L. If $\equiv_L$ is infinite then there is an infinite fooling set for L.

Myhill-Nerode Theorem

Theorem (Myhill-Nerode)

L is regular $\iff$ $\equiv_L$ has a finite number of equivalence classes. If $\equiv_L$ is finite with n equivalence classes then there is a DFA M accepting L with exactly n states and this is the minimum possible.

Corollary

A language L is non-regular if and only if there is an infinite fooling set F for L.

Algorithmic implication: For every DFA M one can find in polynomial time a DFA M' such that $L(M) = L(M')$ and M' has the fewest possible states among all such DFAs.