Bellman-Ford and Dynamic Programming

Lecture 18

Part I

No negative edges: Dijkstra

Dijkstra's Algorithm

```
Initialize for each node v, dist(s, v) = ∞
Initialize X = ∅, dist(s, s) = 0
for i = 1 to |V| do
    Let v be such that dist(s, v) = min_{u∈V−X} dist(s, u)
    X = X ∪ {v}
    for each u in Adj(v) do
        dist(s, u) = min(dist(s, u), dist(s, v) + ℓ(v, u))
```

Priority Queues to maintain *dist* values for faster running time

1. Using heaps and standard priority queues: $O((m+n)\log n)$
2. Best-first-search

Dijkstra's Algorithm using Priority Queues

$Q \leftarrow$ makePQ()
insert(Q, $(s, 0)$)
for each node $u \neq s$ **do**
    insert(Q, (u, ∞))
$X \leftarrow \emptyset$
for $i = 1$ to $|V|$ **do**
    $(v, \mathrm{dist}(s, v)) =$ *extractMin(Q)*
    $X = X \cup \{v\}$
    for each u in $\mathrm{Adj}(v)$ **do**
        decreaseKey$\Big(Q, \big(u, \min\big(\mathrm{dist}(s, u),\ \mathrm{dist}(s, v) + \ell(v, u)\big)\big)\Big)$.

Priority Queue operations:

1. $O(n)$ **insert** operations
2. $O(n)$ **extractMin** operations
3. $O(m)$ **decreaseKey** operations

Implementing Priority Queues via Heaps

Using Heaps

Store elements in a heap based on the key value

1. All operations can be done in $O(\log n)$ time

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Dijkstra's algorithm can be implemented in $O((n+m)\log n)$ time.

Priority Queues: Fibonacci Heaps/Relaxed Heaps

Fibonacci Heaps

1. **extractMin**, **insert**, **delete**, **meld** in $O(\log n)$ time
2. **decreaseKey** in $O(1)$ *amortized* time:

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1. Dijkstra's algorithm can be implemented in $O(n \log n + m)$ time.
2. Data structures are complicated to analyze/implement. Recent work has obtained data structures that are easier to analyze and implement, and perform well in practice. Rank-Pairing Heaps (European Symposium on Algorithms, September 2009!)

Key takeaways of Dijkstra

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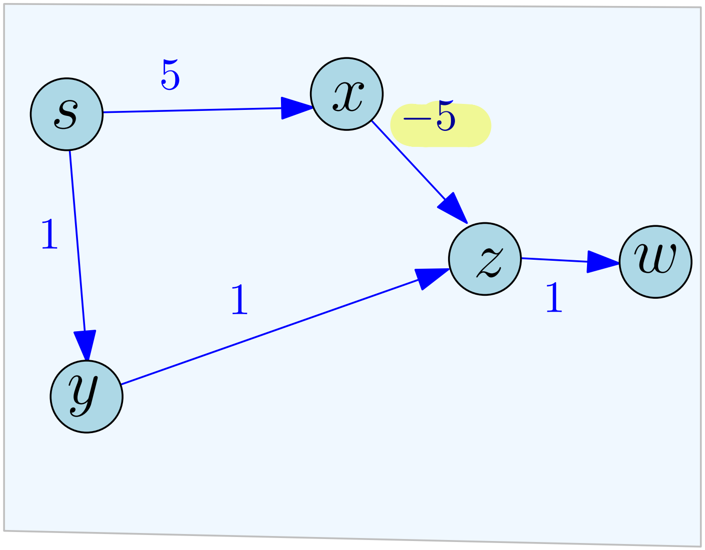
   - $d'(s,u) \geq d(s,u)$
   - $d'(s,v) = \min_{u \in V-X} d'(s,u)$ is the i-th closest node, and $d'(s,v) = d(s,v)$

Part II

Negative Edges: Bellman-Ford

What are the distances computed by Dijkstra's algorithm?

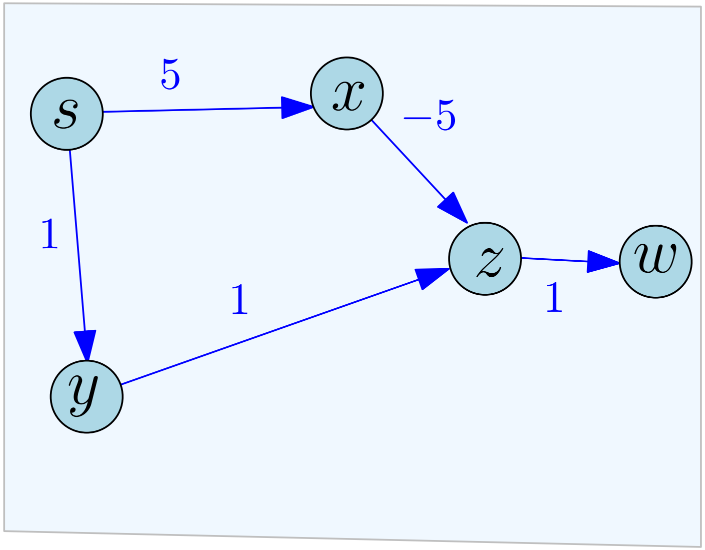


The distance as computed by Dijkstra algorithm starting from s:

- (A) $s = 0$, $x = 5$, $y = 1$, $z = 0$.
- (B) $s = 0$, $x = 1$, $y = 2$, $z = 5$.
- (C) $s = 0$, $x = 5$, $y = 1$, $z = 2$.
- (D) IDK.

Dijkstra's Algorithm and Negative Lengths

With negative length edges, Dijkstra's algorithm can fail



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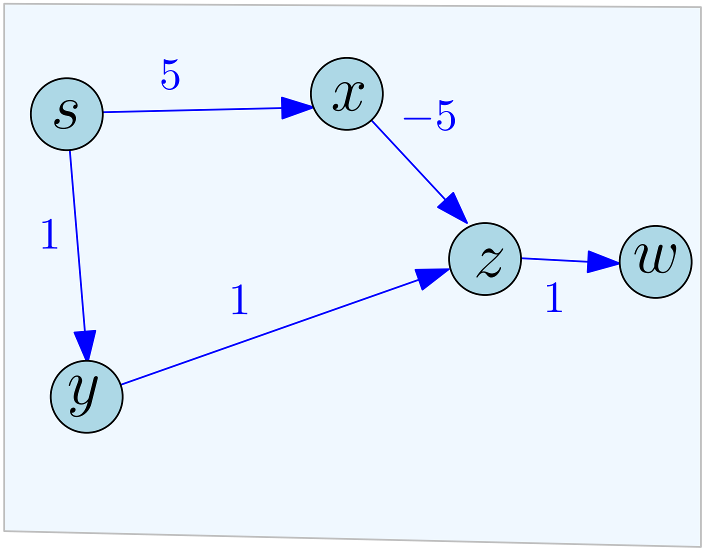

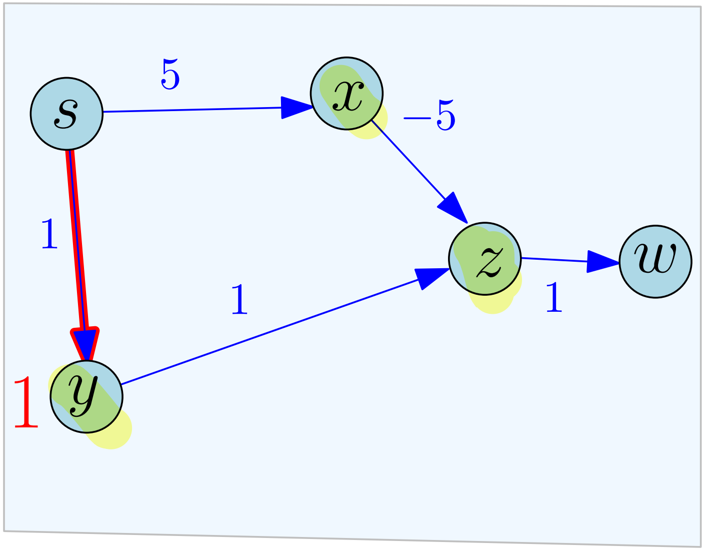

$X = \{s, y\}$

$s \to y \to z \quad d'(s,z) = 2$

$< d'(s,x)$

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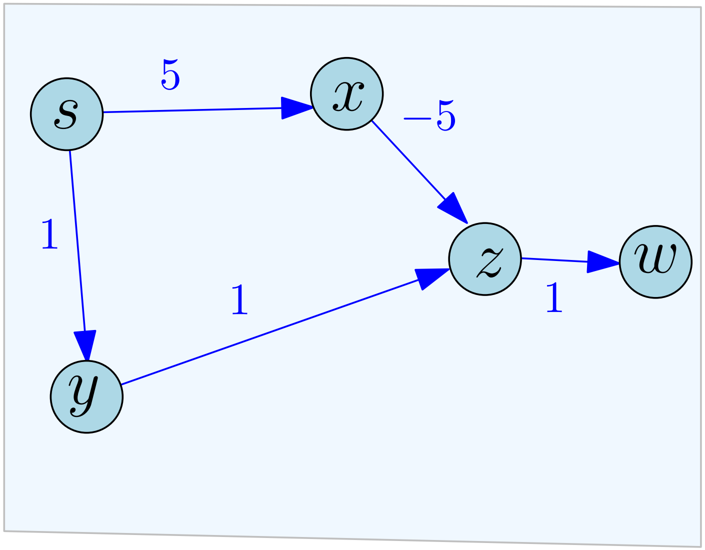

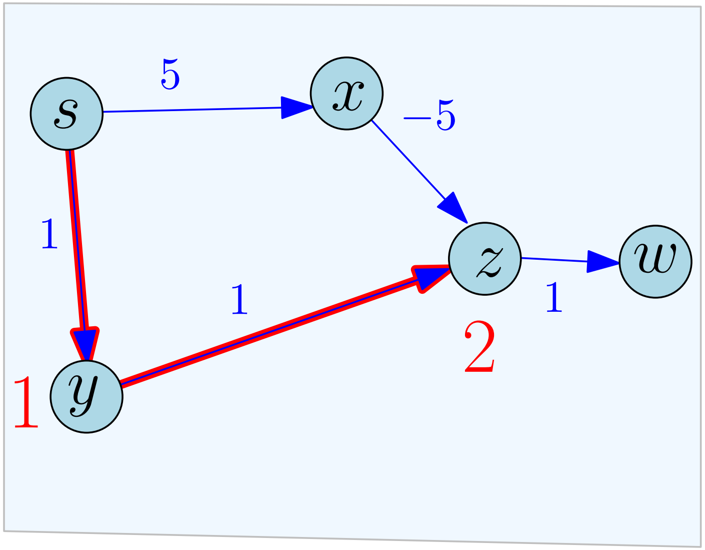

$X = \{ s, y, z \}$

x, w

$d'(s,w) = 3$

$< d'(s,x)$

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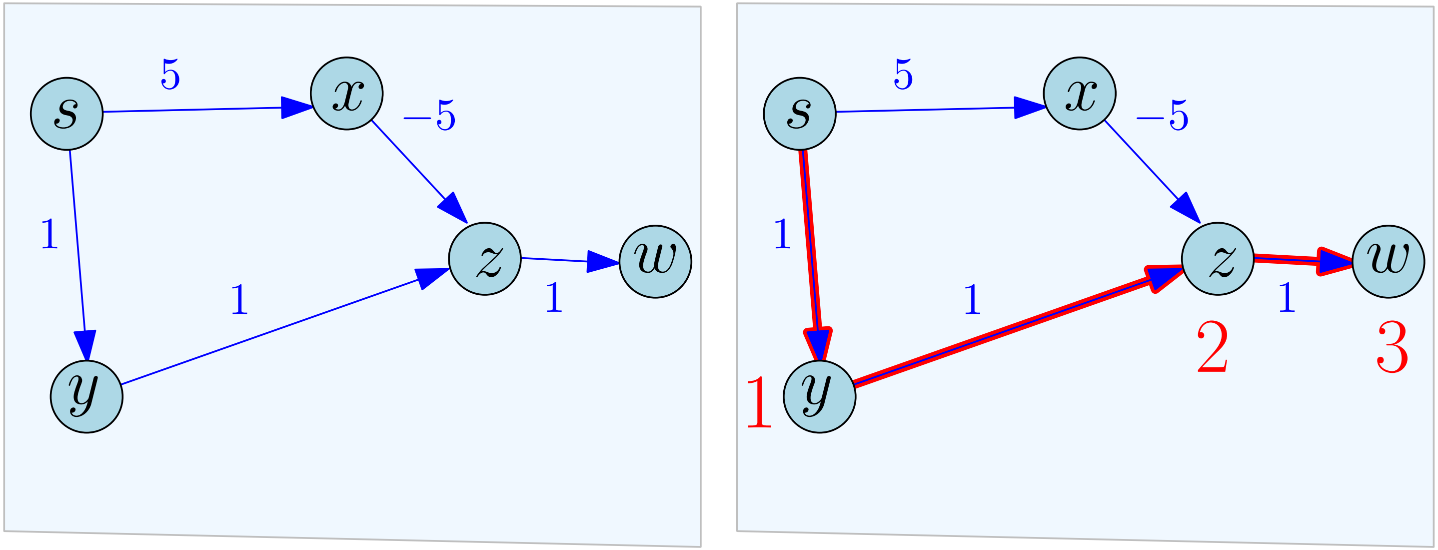

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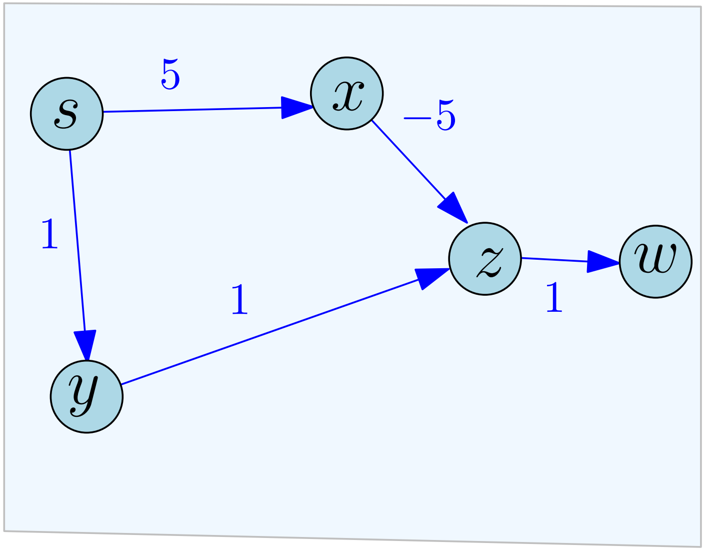

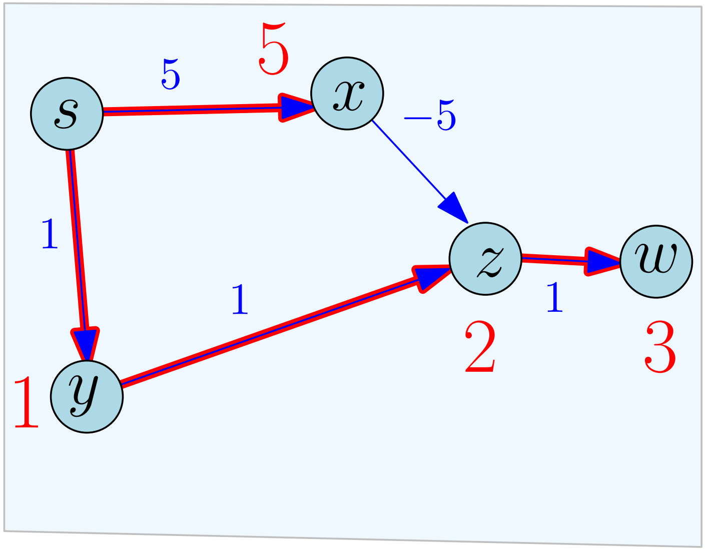

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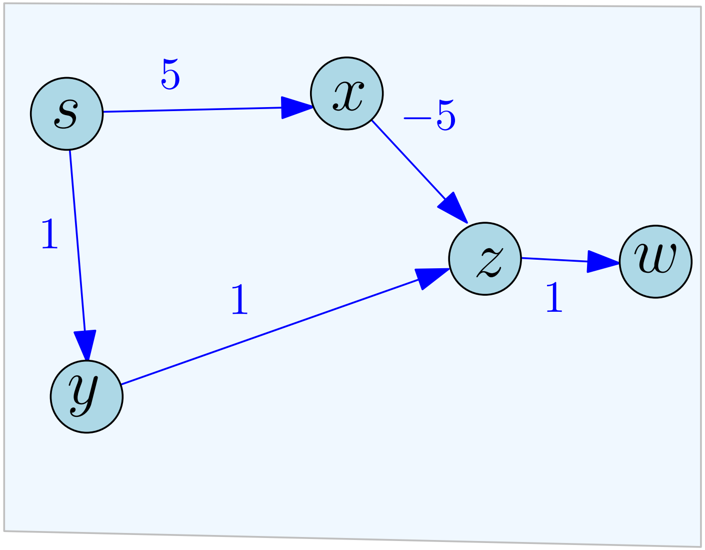

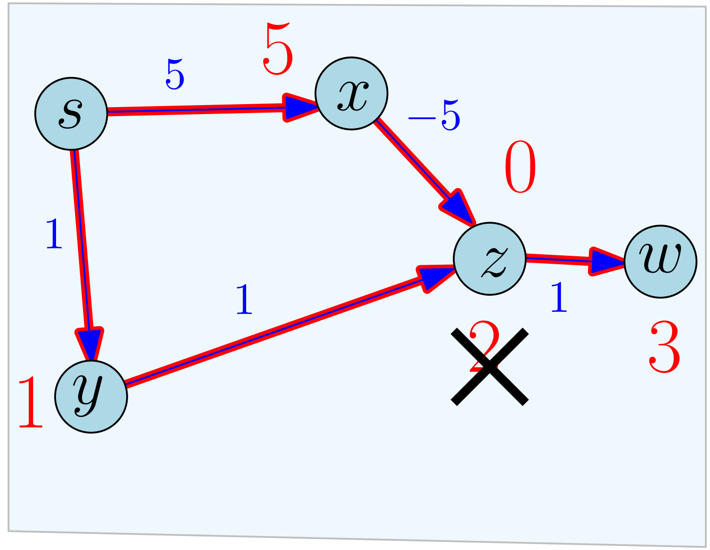

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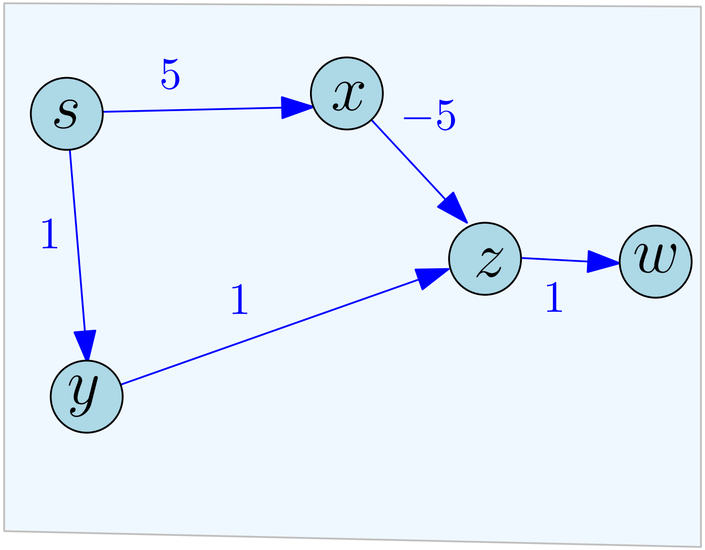

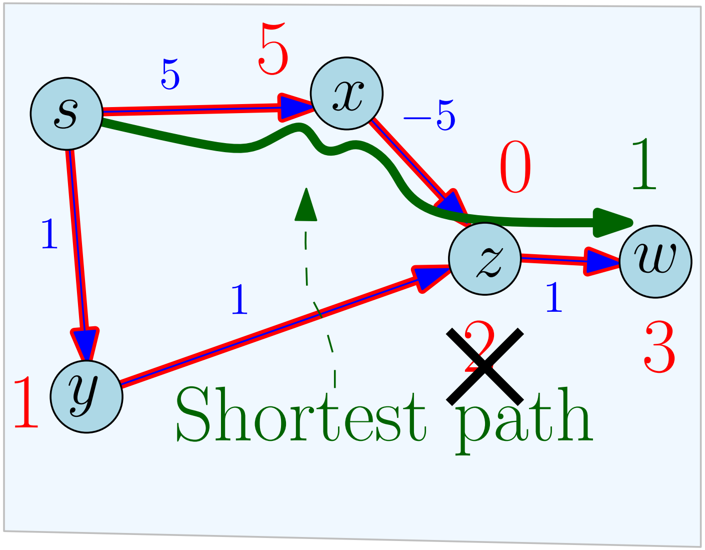

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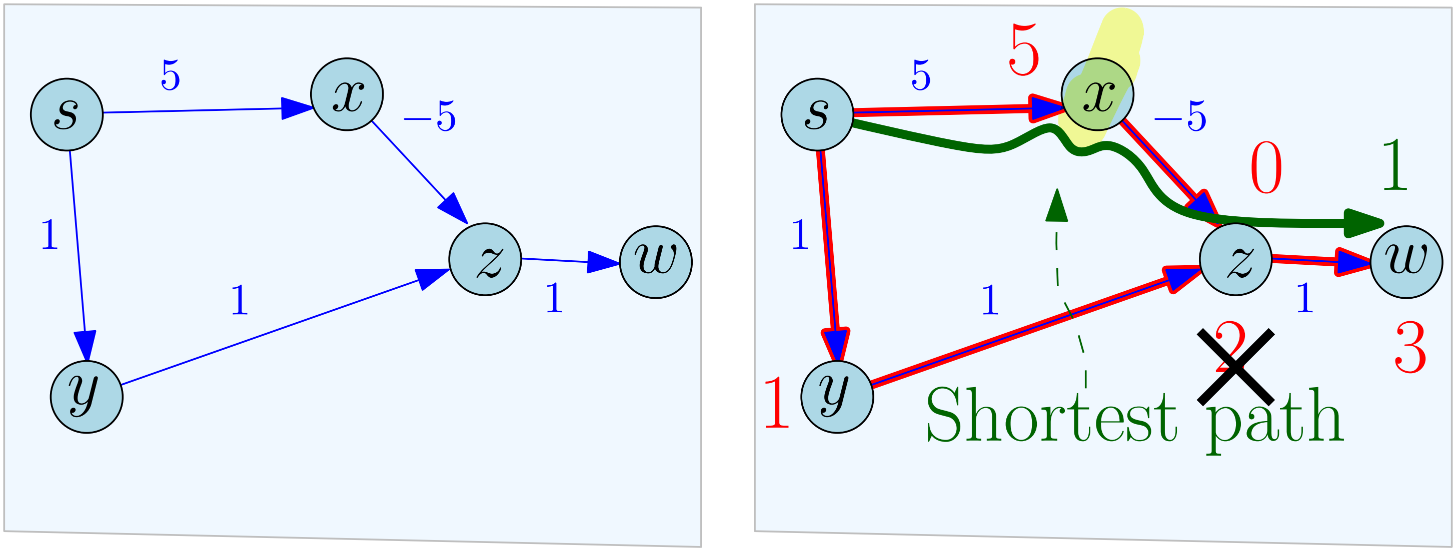

False assumption: Dijkstra's algorithm is based on the assumption that if $s = v_0 \rightarrow v_1 \rightarrow v_2 \ldots \rightarrow v_k$ is a shortest path from s to v_k then $dist(s, v_i) \leq dist(s, v_{i+1})$ for $0 \leq i < k$. Holds true only for non-negative edge lengths.

$d(s, x) > d(s, z)$

Anything we can learn from Dijkstra?

$$d'(s,u) = min\Big(d'(s,u),\ \mathrm{dist}(s,v) + \ell(v,u)\Big)$$

- $d'(s,u) \geq d(s,u)$ still true.

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Solution: Update all edges $|V| - 1$ times!

Bellman-Ford Algorithm

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
    for each v ∈ V do
        for each edge (u, v) ∈ In(v) do
            d(v) = min{d(v), d(u) + ℓ(u, v)}

for each v ∈ V do
    dist(s, v) ← d(v)
```

Running time: $O(mn)$

Part III

Bellman-Ford and DP

Shortest Paths and Recursion

1. Compute the shortest path distance from s to t recursively?
2. What are the smaller sub-problems?

Shortest Paths and Recursion

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2. What are the smaller sub-problems?

Lemma

Let G be a directed graph with arbitrary edge lengths. If $s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_k$ is a shortest path from s to v_k then for $1 \leq i < k$:

1. *$s = v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i$ is a shortest path from s to v_i*

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Sub-problem idea: paths of fewer hops/edges

Hop-based Recursion: Bellman-Ford Algorithm

Single-source problem: fix source s.

$d(v, k)$: shortest path length from s to v using at most k edges.

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Note: $dist(s, v) = d(v, n - 1)$.

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Recursion for $d(v, k)$:

Hop-based Recursion: Bellman-Ford Algorithm

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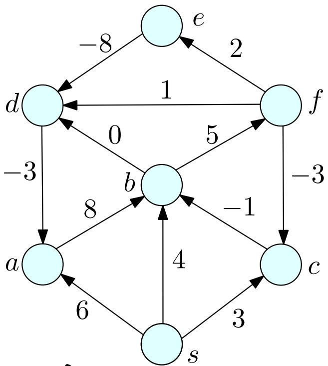
Recursion for $d(v, k)$:

$$d(v,k) = \min \begin{cases} \min_{u \in In(v)} (d(u, k-1) + \ell(u, v)). \\ d(v, k-1) \end{cases}$$

at most k-1 edges

Base case: $d(s, 0) = 0$ and $d(v, 0) = \infty$ for all $v \neq s$.

Example



	0	1	2	3	4	5	6
s	0	0	0	0	0	0	0
a	∞	6	6	1	−1	−1	−2
b	∞	4	2	2	2	2	2
c	∞	3	3	3	3	3	3
d	∞	∞	4	2	2	1	1
e	∞	∞	∞	11	9	9	9
f	∞	∞	9	7	7	7	7

s $\xrightarrow{6}$ a
d $\xrightarrow{-3}$ a

s $\xrightarrow{4}$ b
a $\xrightarrow{8}$ b
c $\xrightarrow{-1}$ b

s $\xrightarrow{3}$ c
f $\xrightarrow{-3}$ c

b $\xrightarrow{0}$ d
f $\xrightarrow{1}$ d
e $\xrightarrow{-8}$ d

f $\xrightarrow{2}$ e

b $\xrightarrow{5}$ f

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Running time:

Bellman-Ford Algorithm

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for $k = 1$ to $n-1$ **do**
    for each $v \in V$ **do**
        $d(v,k) \leftarrow d(v,k-1)$
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for each $v \in V$ **do**
    $\text{dist}(s,v) \leftarrow d(v,n-1)$

Running time: $O(mn)$

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Running time: $O(mn)$ Space:

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Running time: $O(mn)$ Space: $O(n^2)$

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Running time: $O(mn)$ Space: $O(n^2)$

Space can be reduced to $O(n)$.

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for each v ∈ V do
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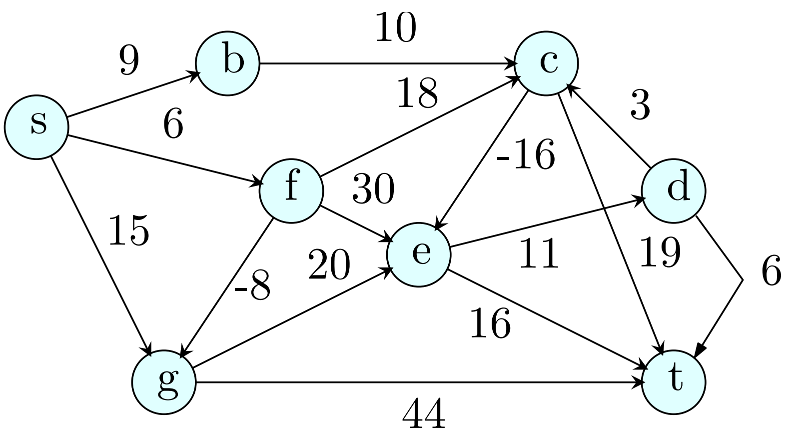
```

Running time: $O(mn)$ Space: $O(n)$

Negative Length Cycles

Definition

A cycle C is a negative length cycle if the sum of the edge lengths of C is negative.



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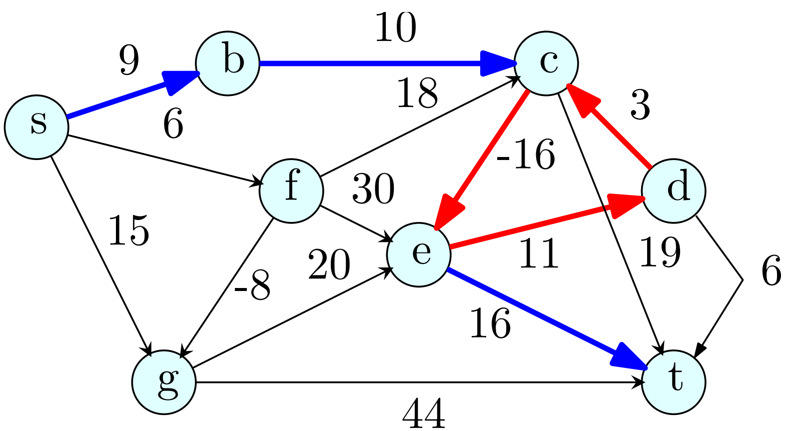

Shortest Paths and Negative Cycles

Given $G = (V, E)$ with edge lengths and s, t. Suppose

1. G has a negative length cycle C, and
2. s can reach C and C can reach t.

Question: What is the shortest **distance** from s to t?

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$-\infty$

Bellman-Ford: Negative Cycle Detection

Check if distances change in iteration n.

```
for each u ∈ V do
    d(u) ← ∞
d(s) ← 0

for k = 1 to n − 1 do
        for each v ∈ V do
            for each edge (u, v) ∈ In(v) do
                d(v) = min{d(v), d(u) + ℓ(u, v)}
(* One more iteration to check if distances change *)
for each v ∈ V do
    for each edge (u, v) ∈ In(v) do
        if (d(v) > d(u) + ℓ(u, v))
            Output ``Negative Cycle''

for each v ∈ V do
        dist(s, v) ← d(v)
```

Negative Cycle Detection

Negative Cycle Detection

Given directed graph G with arbitrary edge lengths, does it have a negative length cycle?

Negative Cycle Detection

Negative Cycle Detection

Given directed graph G with arbitrary edge lengths, does it have a negative length cycle?

1. Bellman-Ford checks whether there is a negative cycle C that is reachable from a specific vertex s. There may negative cycles not reachable from s.
2. Run Bellman-Ford $|V|$ times, once from each node u?

Negative Cycle Detection

1. Add a new node s' and connect it to all nodes of G with zero length edges. Bellman-Ford from s' will find a negative length cycle if there is one. Exercise: why does this work?
2. Negative cycle detection can be done with one Bellman-Ford invocation.