Pre-lecture brain teaser

Is the following language decidable:

$L_{374} = \{\langle M \rangle | L(M) = \{0^{374}\}\}$

CS/ECE-374: Lecture 25 - SAT

Lecturer: Nickvash Kani
Chat moderator: Samir Khan

April 22, 2021

University of Illinois at Urbana-Champaign

Pre-lecture brain teaser

Is the following language decidable:

$L_{374} = \{\langle M \rangle | L(M) = \{0^{374}\}\}$

The Satisfiability Problem (SAT)

Propositional Formulas

Definition
Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

- A *literal* is either a boolean variable x_i or its negation $\neg x_i$.
- A *clause* is a disjunction of literals.
  For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
- A *formula in conjunctive normal form* (CNF) is propositional formula which is a conjunction of clauses
  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.

Definition
Consider a set of boolean variables $x_1, x_2, \ldots x_n$.

- A *literal* is either a boolean variable x_i or its negation $\neg x_i$.
- A *clause* is a disjunction of literals.
  For example, $x_1 \vee x_2 \vee \neg x_4$ is a clause.
- A *formula in conjunctive normal form* (CNF) is propositional formula which is a conjunction of clauses
  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is a CNF formula.
- A formula φ is a 3CNF:
  A CNF formula such that every clause has **exactly** 3 literals.
  - $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3 \vee x_1)$ is a 3CNF formula, but $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is not.

CNF is universal

Every boolean formula $f : \{0,1\}^n \to \{0,1\}$ can be written as a CNF formula.

x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, x_2, \dots, x_6)$	$\overline{x_1} \vee x_2\overline{x_3} \vee x_4 \vee \overline{x_5} \vee x_6$
0	0	0	0	0	0	$f(0, \dots, 0, 0)$	1
0	0	0	0	0	1	$f(0, \dots, 0, 1)$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	0	1	0	0	1	?	1
1	0	1	0	1	0	0	0
1	0	1	0	1	1	?	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
1	1	1	1	1	1	$f(1, \dots, 1)$	1

For every row that f is zero compute corresponding CNF clause.

Take the and ($\bigwedge$) of all the CNF clauses computed

Satisfiability

Problem: SAT

Instance: A CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

Problem: 3SAT

Instance: A 3CNF formula φ.
Question: Is there a truth assignment to the variable of φ such that φ evaluates to true?

SAT
Given a CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

Example

- $(x_1 \vee x_2 \vee \neg x_4) \wedge (x_2 \vee \neg x_3) \wedge x_5$ is satisfiable; take $x_1, x_2, \ldots x_5$ to be all true
- $(x_1 \vee \neg x_2) \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_2) \wedge (x_1 \vee x_2)$ is not satisfiable.

3SAT
Given a 3CNF formula φ, is there a truth assignment to variables such that φ evaluates to true?

(More on **2SAT** in a bit...)

Importance of SAT and 3SAT

- SAT and 3SAT are basic constraint satisfaction problems.
- Many different problems can reduced to them because of the simple yet powerful expressively of logical constraints.
- Arise naturally in many applications involving hardware and software verification and correctness.
- As we will see, it is a fundamental problem in theory of NPCompleteness.

$z = \overline{x}$

Given two bits x, z which of the following SAT formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \vee x) \wedge (z \vee \overline{x})$.

(B) $(z \vee x) \wedge (\overline{z} \vee \overline{x})$.

(C) $(\overline{z} \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (\overline{z} \vee \overline{x})$.

(D) $z \oplus x$.

(E) $(z \vee x) \wedge (\overline{z} \vee \overline{x}) \wedge (z \vee \overline{x}) \wedge (\overline{z} \vee x)$.

$z = \overline{x}$: Solution

Given two bits x, z which of the following SAT formulas is equivalent to the formula $z = \overline{x}$:

(A) $(\overline{z} \lor x) \land (z \lor \overline{x})$.

(B) $(z \lor x) \land (\overline{z} \lor \overline{x})$.

(C) $(\overline{z} \lor x) \land (\overline{z} \lor \overline{x}) \land (\overline{z} \lor \overline{x})$.

(D) $z \oplus x$.

(E) $(z \lor x) \land (\overline{z} \lor \overline{x}) \land (z \lor \overline{x}) \land (\overline{z} \lor x)$.

x	y	$z = \overline{x}$
0	0	0
0	1	1
1	0	1
1	1	0

Given three bits x, y, z which of the following SAT formulas is equivalent to the formula $z = x \wedge y$:

(A) $(\bar{z} \vee x \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(B) $(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(C) $(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(D) $(z \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(E) $(z \vee x \vee y) \wedge (z \vee x \vee \bar{y}) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y}) \wedge$
$(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee x \vee \bar{y}) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (\bar{z} \vee \bar{x} \vee \bar{y})$.

$z = x \wedge y$

Given three bits x, y, z which of the following SAT formulas is equivalent to the formula $z = x \wedge y$:

(A) $(\bar{z} \vee x \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(B) $(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(C) $(\bar{z} \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(D) $(z \vee x \vee y) \wedge (\bar{z} \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y})$.

(E) $(z \vee x \vee y) \wedge (z \vee x \vee \bar{y}) \wedge (z \vee \bar{x} \vee y) \wedge (z \vee \bar{x} \vee \bar{y}) \wedge (\bar{z} \vee x \vee y) \wedge (\bar{z} \vee x \vee \bar{y}) \wedge$

x	y	z	$z = x \wedge y$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Reducing SAT to 3SAT

How SAT is different from 3SAT?
In SAT clauses might have arbitrary length: $1, 2, 3, \ldots$ variables:

$$\left(x \vee y \vee z \vee w \vee u\right) \wedge \left(\neg x \vee \neg y \vee \neg z \vee w \vee u\right) \wedge \left(\neg x\right)$$

In 3SAT every clause must have *exactly* 3 different literals.

How SAT is different from 3SAT?
In SAT clauses might have arbitrary length: $1, 2, 3, \ldots$ variables:

$$\left(x \lor y \lor z \lor w \lor u\right) \land \left(\neg x \lor \neg y \lor \neg z \lor w \lor u\right) \land \left(\neg x\right)$$

In 3SAT every clause must have *exactly* 3 different literals.

To reduce from an instance of SAT to an instance of 3SAT, we must make all clauses to have exactly 3 variables...

Basic idea

- Pad short clauses so they have 3 literals.
- Break long clauses into shorter clauses.
- Repeat the above till we have a 3CNF.

Proof of this in Prof. Har-Peled's async lectures!

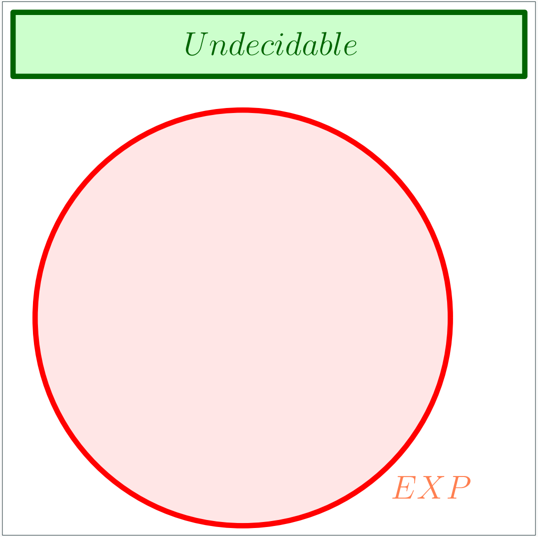
Overview of Complexity Classes

In the beginning...

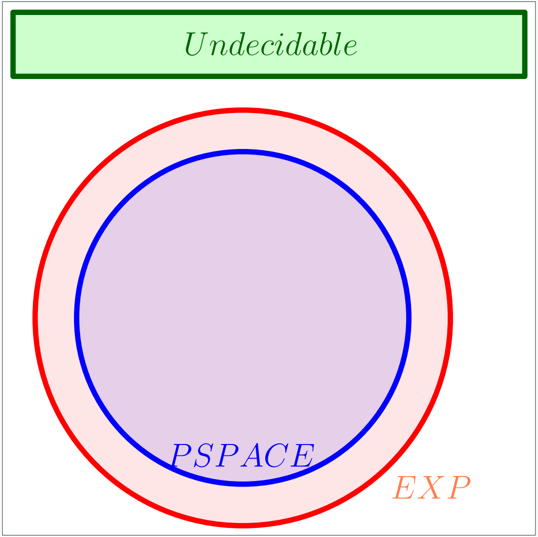
In the beginning...

$\mathit{Undecidable}$

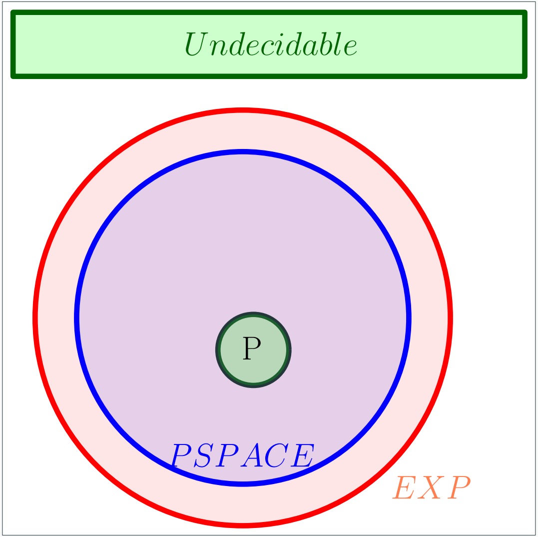
In the beginning...



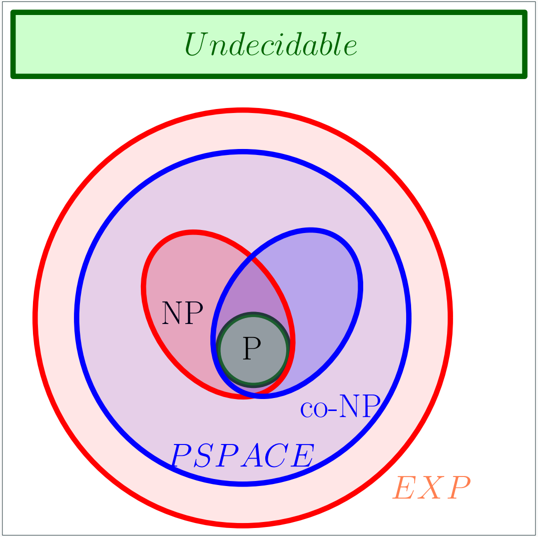
In the beginning...



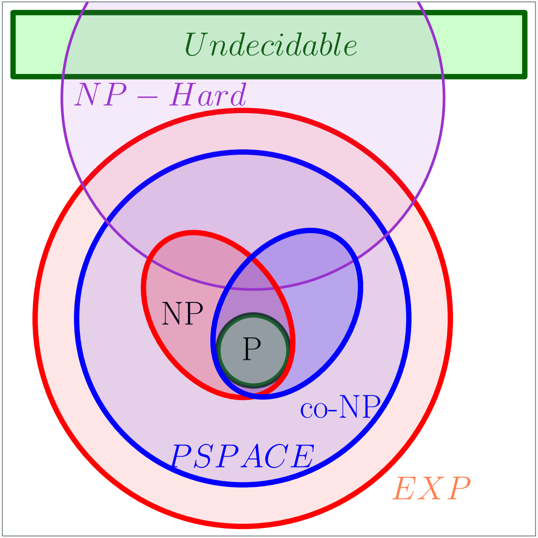
In the beginning...



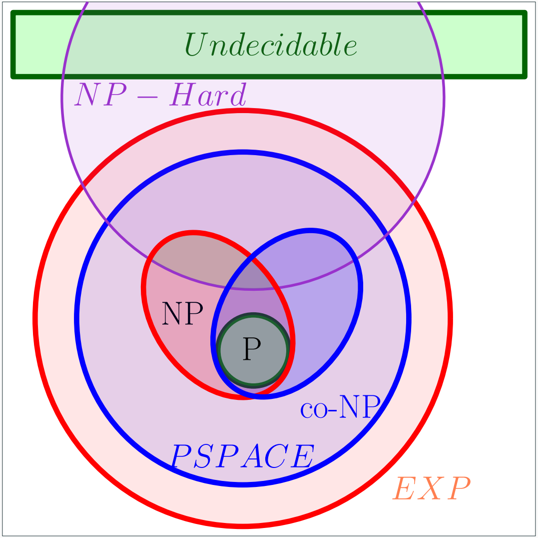
In the beginning...



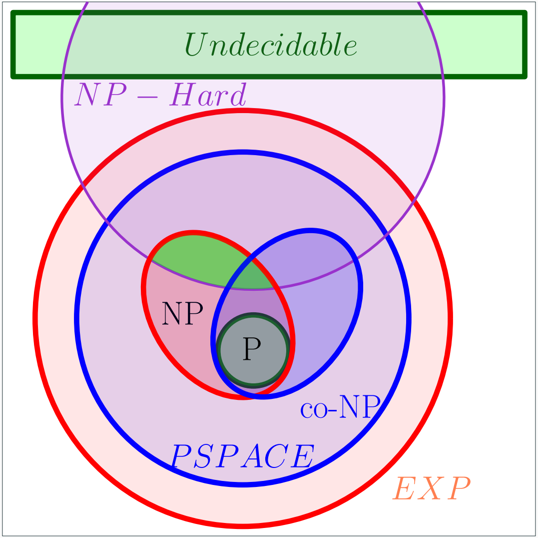
In the beginning...



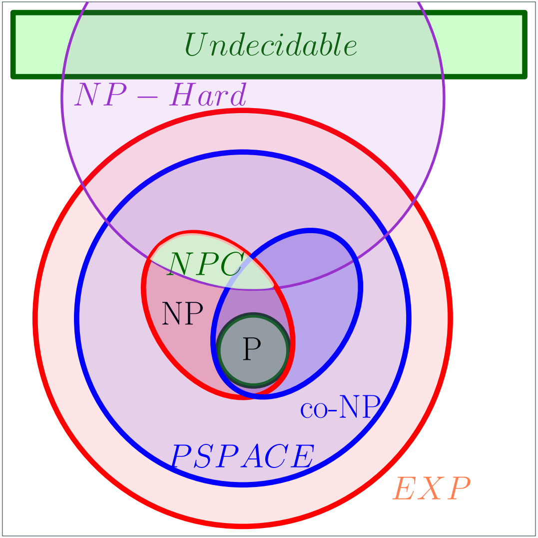
In the beginning...



In the beginning...



In the beginning...



Non-deterministic polynomial time - NP

P and NP and Turing Machines

- P: set of decision problems that have polynomial time algorithms.
- NP: set of decision problems that have polynomial time *non-deterministic* algorithms.

- Many natural problems we would like to solve are in NP.
- Every problem in NP has an exponential time algorithm
- $P \subseteq NP$
- Some problems in NP are in P (example, shortest path problem)

Big Question: Does every problem in NP have an efficient algorithm? Same as asking whether $P = NP$.

Problems with no known deterministic polynomial time algorithms

Problems

- Independent Set
- Vertex Cover
- Set Cover
- SAT

There are of course undecidable problems (no algorithm at all!) but many problems that we want to solve are of similar flavor to the above.

Question: What is common to above problems?

Problems with no known deterministic polynomial time algorithms

Problems

- Independent Set
- Vertex Cover
- Set Cover
- SAT

There are of course undecidable problems (no algorithm at all!) but many problems that we want to solve are of similar flavor to the above.

Question: What is common to above problems?

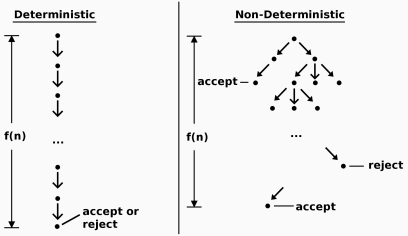
They can all be solved via a non-deterministic computer in polynomial time!

Non-determinism in computing

Non-determinism is a special property of algorithms.

An algorithm that is capable of taking multiple states concurrently. Whenever it reaches a choice, it takes both paths.

If there is a path for the string to be accepted by the machine, then the string is part of the language.



Problems with no known deterministic polynomial time algorithms

Problems

- Independent Set & Vertex Cover - Can build algorithm to check all possible collection of vertices
- Set Cover - Can check all possible collection of sets
- SAT -Can build a non-deterministic algorithm that checks every possible boolean assignment.

But we don't have access to a non-deterministic computer. So how can a deterministic computer verify that a algorithm is in NP?

Above problems share the following feature:

Checkability
For any YES instance I_X of X there is a proof/certificate/solution that is of length poly($|I_X|$) such that given a proof one can efficiently check that I_X is indeed a YES instance.

Efficient Checkability

Above problems share the following feature:

Checkability
For any YES instance I_X of X there is a proof/certificate/solution that is of length poly($|I_X|$) such that given a proof one can efficiently check that I_X is indeed a YES instance.

Examples:

- **SAT** formula φ: proof is a satisfying assignment.
- **Independent Set** in graph G and k: a subset S of vertices.
- **Homework**

Definition
An algorithm $C(\cdot,\cdot)$ is a *certifier* for problem X if the following two conditions hold:

- For every $s \in X$ there is some string t such that $C(s,t) =$ "yes"
- If $s \notin X$, $C(s,t) =$ "no" for every t.

The string s is the problem instance. (Example: particular graph in independent set problem) The string t is called a certificate or proof for s.

Efficient (polynomial time) Certifiers

Definition (Efficient Certifier.)
A certifier C is an *efficient certifier* for problem X if there is a polynomial $p(\cdot)$ such that the following conditions hold:

- For every $s \in X$ there is some string t such that $C(s, t) =$ "yes" and $|t| \leq p(|s|)$.
- If $s \notin X$, $C(s, t) =$ "no" for every t.
- $C(\cdot, \cdot)$ runs in polynomial time.

Example: Independent Set

- Problem: Does $G = (V, E)$ have an independent set of size $\geq k$?
  - Certificate: Set $S \subseteq V$.
  - Certifier: Check $|S| \geq k$ and no pair of vertices in S is connected by an edge.

Example: SAT

- Problem: Does formula φ have a satisfying truth assignment?
  - Certificate: Assignment a of 0/1 values to each variable.
  - Certifier: Check each clause under a and say "yes" if all clauses are true.

Why is it called Nondeterministic Polynomial Time

A certifier is an algorithm $C(I, c)$ with two inputs:

- I: instance.
- c: proof/certificate that the instance is indeed a YES instance of the given problem.

One can think about C as an algorithm for the original problem, if:

- Given I, the algorithm guesses (non-deterministically, and who knows how) a certificate c.
- The algorithm now verifies the certificate c for the instance I.

NP can be equivalently described using Turing machines.

Cook-Levin Theorem

“Hardest” Problems

Question
What is the hardest problem in NP? How do we define it?

Towards a definition

- Hardest problem must be in NP.
- Hardest problem must be at least as “difficult” as every other problem in NP.

Definition
A problem X is said to be **NP-Complete** if

- $X \in NP$, and
- (Hardness) For any $Y \in NP$, $Y \leq_P X$.

Lemma
Suppose X is NP-Complete. Then X can be solved in polynomial time if and only if $P = NP$.

Proof.

$\Rightarrow$ Suppose X can be solved in polynomial time

- Let $Y \in NP$. We know $Y \leq_P X$.
- We showed that if $Y \leq_P X$ and X can be solved in polynomial time, then Y can be solved in polynomial time.
- Thus, every problem $Y \in NP$ is such that $Y \in P$; $NP \subseteq P$.
- Since $P \subseteq NP$, we have $P = NP$.

$\Leftarrow$ Since $P = NP$, and $X \in NP$, we have a polynomial time algorithm for X. $\square$

NP-Hard Problems

Definition
A problem Y is said to be NP-Hard if

- (Hardness) For any $X \in NP$, we have that $X \leq_P Y$.

An NP-Hard problem need not be in NP!

Example: Halting problem is NP-Hard (why?) but not NP-Complete.

Consequences of proving NP-Completeness

If X is NP-Complete

- Since we believe $P \neq NP$,
- and solving X implies $P = NP$.

X is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for X.

Consequences of proving NP-Completeness

If X is NP-Complete

- Since we believe $P \neq NP$,
- and solving X implies $P = NP$.

X is unlikely to be efficiently solvable.

At the very least, many smart people before you have failed to find an efficient algorithm for X.

(This is proof by mob opinion — take with a grain of salt.)

Question
Are there any problems that are NP-Complete?

Answer
Yes! Many, many problems are NP-Complete.

Cook-Levin Theorem

Theorem (Cook-Levin)
SAT is NP-Complete.

Cook-Levin Theorem

Theorem (Cook-Levin)
SAT is NP-Complete.

Need to show

- SAT is in NP.
- *every* NP problem X reduces to SAT.

Steve Cook won the Turing award for his theorem.

Proving that a problem X is NP-Complete

To prove X is NP-Complete, show

- Show that X is in NP.
- Give a polynomial-time reduction *from* a known NP-Complete problem such as SAT *to* X

Proving that a problem X is NP-Complete

To prove X is NP-Complete, show

- Show that X is in NP.
- Give a polynomial-time reduction *from* a known NP-Complete problem such as SAT *to* X

SAT $\leq_P X$ implies that every NP problem $Y \leq_P X$. Why?

Proving that a problem X is NP-Complete

To prove X is NP-Complete, show

- Show that X is in NP.
- Give a polynomial-time reduction *from* a known NP-Complete problem such as **SAT** *to* X

SAT $\leq_P X$ implies that every NP problem $Y \leq_P X$. Why? Transitivity of reductions:

$Y \leq_P SAT$ and $SAT \leq_P X$ and hence $Y \leq_P X$.

3-SAT is NP-Complete

- 3-SAT is in *NP*
- SAT $\leq_P$ 3-SAT as we saw

NP-Completeness via Reductions

- SAT is NP-Complete due to Cook-Levin theorem
- SAT $\leq_P$ 3-SAT
- 3-SAT $\leq_P$ Independent Set
- Independent Set $\leq_P$ Vertex Cover
- Independent Set $\leq_P$ Clique
- 3-SAT $\leq_P$ 3-Color
- 3-SAT $\leq_P$ Hamiltonian Cycle

NP-Completeness via Reductions

- SAT is NP-Complete due to Cook-Levin theorem
- SAT $\leq_P$ 3-SAT
- 3-SAT $\leq_P$ Independent Set
- Independent Set $\leq_P$ Vertex Cover
- Independent Set $\leq_P$ Clique
- 3-SAT $\leq_P$ 3-Color
- 3-SAT $\leq_P$ Hamiltonian Cycle

Hundreds and thousands of different problems from many areas of science and engineering have been shown to be NP-Complete.

A surprisingly frequent phenomenon!

Reducing 3-SAT to Independent Set

Independent Set

Problem: Independent Set

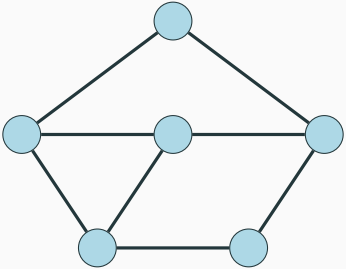
Instance: A graph G, integer k.

Question: Is there an independent set in G of size k?

Independent Set

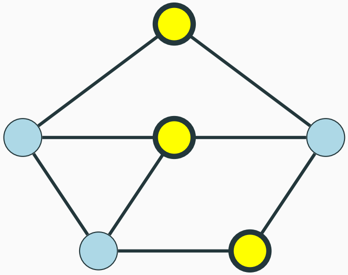
Problem: Independent Set

> **Instance**: A graph G, integer k.
> **Question**: Is there an independent set in G of size k?



Problem: Independent Set

Instance: A graph G, integer k.
Question: Is there an independent set in G of size k?



Interpreting 3SAT

There are two ways to think about 3SAT

- Find a way to assign 0/1 (false/true) to the variables such that the formula evaluates to true, that is each clause evaluates to true.
- Pick a literal from each clause and find a truth assignment to make all of them true. You will fail if two of the literals you pick are in conflict, i.e., you pick x_i and $\neg x_i$

We will take the second view of 3SAT to construct the reduction.

The Reduction

- G_φ will have one vertex for each literal in a clause
- 2- Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
- 4- Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
- 5- Take k to be the number of clauses

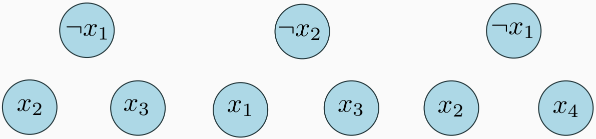


Figure 1: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

The Reduction

- G_φ will have one vertex for each literal in a clause
- 2- Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
- 4- Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
- 5- Take k to be the number of clauses

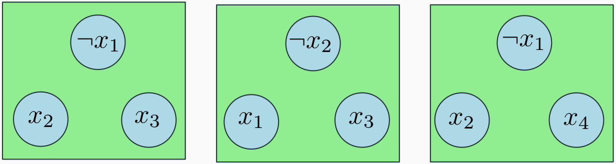


Figure 1: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

The Reduction

- G_φ will have one vertex for each literal in a clause
- 2- Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
- 4- Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
- 5- Take k to be the number of clauses

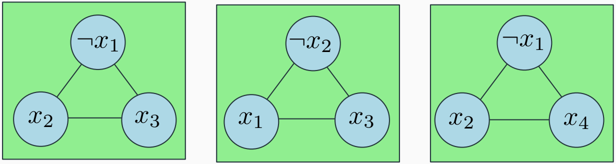


Figure 1: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

- G_φ will have one vertex for each literal in a clause
- 2- Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
- 4- Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
- 5- Take k to be the number of clauses

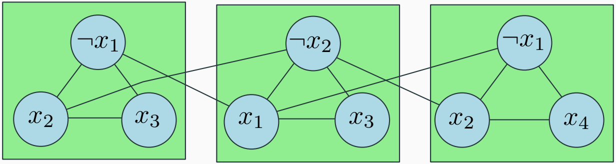


Figure 1: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

The Reduction

- G_φ will have one vertex for each literal in a clause
- 2- Connect the 3 literals in a clause to form a triangle; the independent set will pick at most one vertex from each clause, which will correspond to the literal to be set to true
- 4- Connect 2 vertices if they label complementary literals; this ensures that the literals corresponding to the independent set do not have a conflict
- 5- Take k to be the number of clauses

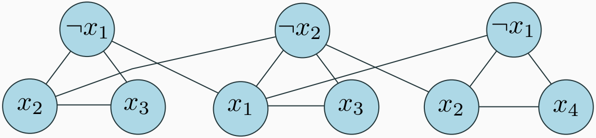


Figure 1: Graph for $\varphi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee x_4)$

Lemma
φ is satisfiable iff G_φ has an independent set of size k (= number of clauses in φ).

Proof.

$\Rightarrow$ Let a be the truth assignment satisfying φ

- 2- Pick one of the vertices, corresponding to true literals under a, from each triangle. This is an independent set of the appropriate size. Why? $\square$

Correctness (contd)

Lemma
φ is satisfiable iff G_φ has an independent set of size k ($=$ number of clauses in φ).

Proof.

$\Leftarrow$ Let S be an independent set of size k

- S must contain *exactly* one vertex from each clause triangle
- S cannot contain vertices labeled by conflicting literals
- Thus, it is possible to obtain a truth assignment that makes in the literals in S true; such an assignment satisfies one literal in every clause $\square$