Programming Languages and Compilers (CS 421)

Sasa Misailovic
4110 SC, UIUC

https://courses.engr.illinois.edu/cs421/fa2017/CS421A

Based in part on slides by Mattox Beckman, as updated by Vikram Adve, Gul Agha, and Elsa L Gunter

Structural Recursion

- **Functions on recursive datatypes (eg lists) tend to be recursive**
- Recursion over recursive datatypes generally by **structural recursion**
  - Recursive calls made to components of structure of the same recursive type
  - Base cases of recursive types stop the recursion of the function

Structural Recursion : List Example

```
# let rec length list = match list with
    [ ] -> 0   (* Nil case *)
  | x :: xs -> 1 + length xs;; (* Cons case *)
val length : 'a list -> int = <fun>

# length [5; 4; 3; 2];;
- : int = 4
```

- Nil case [ ] is base case
- Cons case recurses on component list xs

Forward Recursion

- In Structural Recursion, split input into components and (eventually) recurse on components
- Forward Recursion form of Structural Recursion
- In forward recursion, **first call the function recursively** on all recursive components, and then build final result from partial results
- Wait until whole structure has been traversed to start building answer

Forward Recursion: Examples

```
# let rec double_up list =
    match list with
        [ ] -> [ ]
      | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>

# let rec poor_rev list =
  match list with
      [] -> []
    | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```


Question

- How do you write length with forward recursion?

```
let rec length l =
```

Question

- How do you write length with forward recursion?

```
let rec length l =
    match l with [] ->
     | (a :: bs) ->
```


Question

- How do you write length with forward recursion?

```
let rec length l =
    match l with [] ->
     | (a :: bs) ->     length bs
```


Question

- How do you write length with forward recursion?

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```


Question

- How do you write length with forward recursion?

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> let t = length bs
                    in 1 + t
```


Functions Over Lists

```
# let rec double_up list =
    match list
    with [ ] -> [ ]  (* pattern before ->,
                         expression after *)
       | (x :: xs) -> (x :: x :: double_up xs);;
val double_up : 'a list -> 'a list = <fun>
```

```
# let fib5_2 = double_up fib5;;
val fib5_2 : int list = [8; 8; 5; 5; 3; 3; 2; 2;
  1; 1; 1; 1]
```


Functions Over Lists

```
# let rec poor_rev list =
  match list
  with [] -> []
     | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

```
# poor_rev silly;;
- : string list = ["there"; "there"; "hi"; "hi"]
```

Your Turn

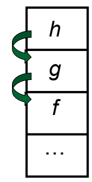
- Write a function odd_count fr : int list -> int such that it returns the number of odd integers found in the input list. The function is required to use (only) forward recursion (no other form of recursion).

```
# let rec odd_count_fr l =


# odd_count_fr [1;2;3];;
- : int = 2
```


An Important Optimization

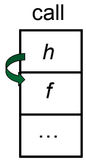
Normal call



- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?
  - `let f x = (g x) + 1`
  - `let g x = h (x+1)`
  - `let h x = …`

An Important Optimization

Tail call



- When a function call is made, the return address needs to be saved to the stack so we know to where to return when the call is finished
- What if *f* calls *g* and *g* calls *h*, but calling *h* is the last thing *g* does (a *tail call*)?
- Then *h* can return directly to *f* instead of *g*

Tail Recursion

- A recursive program is tail recursive if all recursive calls are tail calls
- Tail recursive programs may be optimized to be implemented as loops, thus removing the function call overhead for the recursive calls
- Tail recursion generally requires extra "accumulator" arguments to pass partial results
  - May require an auxiliary function

Example of Tail Recursion

```
# let rec prod l =
    match l with [] -> 1
    | (x :: rem) -> x * prod rem;;
val prod : int list -> int = <fun>

# let prod list =
    let rec prod_aux l acc =
        match l with [] -> acc
        | (y :: rest) -> prod_aux rest (acc * y)
(* Uses associativity of multiplication *)
    in prod_aux list 1;;
 val prod : int list -> int = <fun>
```


Question

- How do you write length with tail recursion?

```
let length l =
```

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =



in
```


Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] ->
     | (a :: bs) ->
in
```


Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) ->
in
```


Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux
in
```


Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs
in
```


Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs (n + 1)
in
```

Question

- How do you write length with tail recursion?

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
     | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0
```


Your Turn

- Write a function odd_count_tr : int list -> int such that it returns the number of odd integers found in the input list. The function is required to use (only) tail recursion (no other form of recursion).

```
# let rec odd_count_tr l =


# odd_count_tr [1;2;3];;
- : int = 2
```


Mapping Recursion

- One common form of structural recursion applies a function to each element in the structure

```
# let rec doubleList list = match list with
     [ ] -> [ ]
   | x::xs -> 2 * x :: doubleList xs;;
val doubleList : int list -> int list = <fun>

# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```


Mapping Functions Over Lists

```
# let rec map f list =
  match list with
    [] -> []
  | (h::t) -> (f h) :: (map f t);;
val map : ('a -> 'b)-> 'a list-> 'b list = <fun>

# map plus_two fib5;;
- : int list = [10; 7; 5; 4; 3; 3]

# map (fun x -> x - 1) fib6;;
: int list = [12; 7; 4; 2; 1; 0; 0]
```


Mapping Recursion

- Can use the higher-order recursive map function instead of direct recursion

```
# let doubleList list =
    List.map (fun x -> 2 * x) list;;
val doubleList : int list -> int list = <fun>

# doubleList [2;3;4];;
- : int list = [4; 6; 8]
```

- Same function, but no rec

Your turn now

Write a function

`make_app : ((‘a -> ‘b) * ‘a) list -> ‘b list`

that takes a list of function – input pairs and gives the result of applying each function to its argument. Use map, no explicit recursion.

```
let make_app lst =
```

Folding Recursion

- Another common form "folds" an operation over the elements of the structure

```
# let rec multList list = match list with
    [ ] -> 1
  | x::xs -> x * multList xs;;
val multList : int list -> int = <fun>

# multList [2;4;6];;
- : int = 48
```

- Computes (2 * (4 * (6 * 1)))

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumlist list = match list with
    [ ] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
    [ ] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```


Folding Functions over Lists

How are the following functions similar?

```
# let rec sumlist list = match list with
    [ ] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
    [ ] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```

Base Case

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumlist list = match list with
    [ ] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
    [ ] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```

Recursive Call

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumlist list = match list with
    [ ] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
    [ ] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```

Head Element

Folding Functions over Lists

How are the following functions similar?

```
# let rec sumlist list = match list with
    [ ] -> 0
  | x::xs -> x + sumlist xs;;

# sumlist [2;3;4];;
- : int = 9

# let rec prodlist list = match list with
    [ ] -> 1
  | x::xs -> x * prodlist xs;;

# prodlist [2;3;4];;
- : int = 24
```

Combining Operator

Recursing over lists

```
# let rec fold_right f list b =
  match list with
    [] -> b
  | (x :: xs) -> f x (fold_right f xs b);;

# fold_right
    (fun val init -> val + init)
    [1; 2; 3]
    0;;
- : int = 6
```


Recursing over lists

```
# let rec fold_right f list b =
  match list with
    [] -> b
  | (x :: xs) -> f x (fold_right f xs b);;

# fold_right
    (fun s -> fun () -> print_string s)
    ["hi"; "there"]
    ();;
therehi- : unit = ()
```


Folding Recursion

- multList folds to the right
- Same as:

```
# let multList list =
    List.fold_right
       (fun x -> fun p -> x * p)
       list 1;;
val multList : int list -> int = <fun>

# multList [2;4;6];;
- : int = 48
```


Encoding Recursion with Fold

```
# let rec append list1 list2 = match list1 with
  [ ] -> list2 | x::xs -> x :: append xs list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
```

Base Case | Operation | Recursive call

```
# let append list1 list2 =
   fold_right (fun x y -> x :: y) list1 list2;;
val append : 'a list -> 'a list -> 'a list = <fun>
# append [1;2;3] [4;5;6];;
 - : int list = [1; 2; 3; 4; 5; 6]
```


Question

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

- How do you write length with fold_right, but no explicit recursion?

Question

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

- How do you write length with fold_right, but no explicit recursion?

```
let length list =
   List.fold_right (fun x -> fun n ->  n + 1)
                   list 0
```

Question

```
let rec length l =
    match l with [] -> 0
     | (a :: bs) -> 1 + length bs
```

- How do you write length with fold_right, but no explicit recursion?

```
let length list =
   List.fold_right (fun x -> fun n ->  n + 1)
                   list 0
```

Can you write fold_right (or fold_left) with just map? How, or why not?

Iterating over lists

```
# let rec fold_left f a list =
  match list with
    [] -> a
  | (x :: xs) -> fold_left f (f a x) xs;;
val fold_left : ('a -> 'b -> 'a) -> 'a -> 'b list
  -> 'a = <fun>

# fold_left
   (fun () -> print_string)
   ()
   ["hi"; "there"];;
hithere- : unit = ()
```


Encoding Tail Recursion with fold_left

```
# let prod list = let rec prod_aux l acc =
        match l with
          [] -> acc
        | (y :: rest) -> prod_aux rest (acc * y)
       in prod_aux list 1;;
```

Init Acc Value

Recursive Call

Operation

```
# let prod list =
   List.fold_left (fun acc y -> acc * y) 1 list;;

# prod [4;5;6];;
 - : int =120
```


Question

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
      | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0
```

- How do you write length with fold_left, but no explicit recursion?

Question

```
let length l =
    let rec length_aux list n =
    match list with [] -> n
      | (a :: bs) -> length_aux bs (n + 1)
in length_aux l 0
```

- How do you write length with fold_left, but no explicit recursion?

```
let length list =
 List.fold_left (fun n -> fun x -> n + 1)
                 0 list
```


Folding

```
# let rec fold_left f a list = match list with
    [] -> a
  | (x :: xs) -> fold_left f (f a x) xs;;
```

fold_left f a [x_1; x_2;…;x_n] = f(…(f (f a x_1) x_2)…)x_n

```
# let rec fold_right f list b = match list with
    [] -> b
  | (x :: xs) -> f x (fold_right f xs b);;
```

fold_right f [x_1; x_2;…;x_n] b = f x_1(f x_2 (…(f x_n b)…))

Recall

```
# let rec poor_rev list = match list with
      [] -> []
    | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```

- What is its running time?

Quadratic Time

- Each step of the recursion takes time proportional to input
- Each step of the recursion makes only one recursive call.
- List example:

```
# let rec poor_rev list = match list with
      [] -> []
    | (x::xs) -> poor_rev xs @ [x];;
val poor_rev : 'a list -> 'a list = <fun>
```


Comparison

- poor_rev [1,2,3] =
- (poor_rev [2,3]) @ [1] =
- ((poor_rev [3]) @ [2]) @ [1] =
- (((poor_rev [ ]) @ [3]) @ [2]) @ [1] =
- (([ ] @ [3]) @ [2]) @ [1]) =
- ([3] @ [2]) @ [1] =
- (3:: ([ ] @ [2])) @ [1] =
- [3,2] @ [1] =
- 3 :: ([2] @ [1]) =
- 3 :: (2:: ([ ] @ [1])) = [3, 2, 1]

Tail Recursion - Example

```
# let rec rev_aux list revlist =
  match list with
    [ ] -> revlist
  | x :: xs -> rev_aux xs (x::revlist);;
val rev_aux : 'a list -> 'a list -> 'a list =
  <fun>

# let rev list = rev_aux list [ ];;
val rev : 'a list -> 'a list = <fun>
```

- What is its running time?

Comparison

- rev [1,2,3] =
- rev_aux [1,2,3] [ ] =
- rev_aux [2,3] [1] =
- rev_aux [3] [2,1] =
- rev_aux [ ] [3,2,1] = [3,2,1]

Folding - Tail Recursion

```
# let rec rev_aux list revlist =
  match list with
    [ ] -> revlist
  | x :: xs -> rev_aux xs (x::revlist);;
# let rev list = rev_aux list [ ];;

# let rev list =
      fold_left
        (fun l -> fun x -> x :: l)  (* comb op *)
          []                   (* accumulator cell *)
          list
```

Folding

- Can replace recursion by **fold_right** in any **forward primitive** recursive definition
  - Primitive recursive means it only recurses on immediate subcomponents of recursive data structure
- Can replace recursion by **fold_left** in any **tail primitive** recursive definition

Example of Tail Recursion

```
# let rec app fl x =
    match fl with [] -> x
     | (f :: rem_fs) -> f (app rem_fs x);;
val app : ('a -> 'a) list -> 'a -> 'a = <fun>

# let app fs x =
    let rec app_aux fl acc =
        match fl with [] -> acc
        | (f :: rem_fs) -> app_aux rem_fs
                                   (fun z -> acc (f z))
    in app_aux fs (fun y -> y) x;;
val app : ('a -> 'a) list -> 'a -> 'a = <fun>
```


Continuation Passing Style

- A programming technique for all forms of "non-local" control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls
- Essentially it's a higher-order function version of GOTO

Continuations

- Idea: Use functions to represent the control flow of a program
- Method: Each procedure takes a function as an argument to which to pass its result; outer procedure "returns" no result
- Function receiving the result called a continuation
- Continuation acts as "accumulator" for work still to be done

Continuation Passing Style

- Writing procedures so that they take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)

Example

- Simple reporting continuation:

```
# let report x = (print_int x;
                  print_newline( ) );;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```
# let plusk a b k = k (a + b)
val plusk : int -> int -> (int -> 'a) -> 'a
  = <fun>
# plusk 20 22 report;;
42
- : unit = ()
```

Example of Tail Recursion & CSP

```
# let app fs x =
    let rec app_aux fl acc=
        match fl with
          [] -> acc
        | (f :: rem_fs) -> app_aux rem_fs
                                 (fun z -> acc (f z))
    in app_aux fs (fun y -> y) x;;
val app : ('a -> 'a) list -> 'a -> 'a = <fun>

# let rec appk fl x k =
    match fl with
      [] -> k x
    | (f :: rem_fs) -> appk rem_fs x (fun z -> k (f z));;
hval appk : ('a -> 'a) list -> 'a -> ('a -> 'b) -> 'b
```


Example of Tail Recursion & CSP

```
# let rec appk fl x k =
    match fl with
      [] -> k x
    | (f :: rem_fs) -> appk rem_fs x (fun z -> k (f z));;

# appk [(fun x->x+1); (fun x -> x*5)] 2 (fun x->x);;
- : int = 11
```


Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.
- A formalization of non-local control flow in denotational semantics
- Possible intermediate state in compiling functional code

Terms

- A function is in **Direct Style** when it returns its result back to the caller.
- A **Tail Call** occurs when a function returns the result of another function call without any more computations (e.g. tail recursion)
- A function is in **Continuation Passing Style** when it passes its result to another function.
  - Instead of returning the result to the caller, we pass it forward to another function.

Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.
- A formalization of non-local control flow in denotational semantics
- Possible intermediate state in compiling functional code

Example

- Simple reporting continuation:

```
# let report x = (print_int x;
                  print_newline( ) );;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```
# let plusk a b k = k (a + b)
val plusk : int -> int -> (int -> 'a) -> 'a
  = <fun>
# plusk 20 22 report;;
42
- : unit = ()
```

Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation
- Examples:

```
# let subk x y k = k(x + y);;
val subk : int -> int -> (int -> 'a) -> 'a = <fun>
# let eqk x y k = k(x = y);;
val eqk : 'a -> 'a -> (bool -> 'b) -> 'b = <fun>
# let timesk x y k = k(x * y);;
val timesk : int -> int -> (int -> 'a) -> 'a =
  <fun>
```


Nesting Continuations

```
# let add_three x y z = x + y + z;;
val add_three : int -> int -> int -> int = <fun>

# let add_three x y z = let p = x + y in p + z;;
val add_three : int -> int -> int -> int = <fun>

# let add_three_k x y z k =
   addk x y (fun p -> addk p z k );;
val add_three_k : int -> int -> int -> (int -> 'a
  -> 'a = <fun>
```