Programming Languages and Compilers (CS 421)

Sasa Misailovic

4110 SC, UIUC

https://courses.engr.illinois.edu/cs421/fa2017/CS421A

Based on slides by Elsa Gunter, which were inspired by earlier slides by Mattox Beckman, Vikram Adve, and Gul Agha

Two Problems

- **Type checking**
  - Question: Does exp. e have type τ in env Γ?
  - Answer: Yes / No
  - Method: Type derivation
- **Typability**
  - Question Does exp. e have some type in env. Γ? If so, what is it?
  - Answer: Type τ / error
  - Method: Type inference

Type Inference - Outline

- Begin by assigning a **type variable** as the type of the **whole expression**
- **Decompose the expression** into component expressions
- **Use typing rules to generate constraints** on components and whole
- **Recursively find substitution** that solves typing judgment of first subcomponent
- **Apply substitution to next subcomponent** and find substitution solving it; compose with first, etc.
- **Apply composition of all substitutions** to original type variable to get answer

Type Inference - Example

- What type can we give to

(fun x -> fun f -> f (f x))

- Start with a type variable and then look at the way the term is constructed

Type Inference - Example

$$\frac{\{x : \tau_1\} + \Gamma \mid\!- e : \tau_2}{\Gamma \mid\!- \text{fun } x \text{ -> } e : \tau_1 \to \tau_2}$$

- First approximate: **Give type to full expr**

{ }|- (fun x -> fun f -> f (f x)) : α

- Second approximate: **use fun rule**

$$\frac{\{x : \beta\} \mid\!- (\text{fun f -> f (f x)}) : \gamma}{\{\ \} \mid\!- (\text{fun x -> fun f -> f (f x)}) : \alpha}$$

- Remember constraint $\alpha \equiv (\beta \to \gamma)$

Type Inference - Example

$$\frac{\{x : \tau_1\} + \Gamma \mid\!- e : \tau_2}{\Gamma \mid\!- \text{fun } x \text{ -> } e : \tau_1 \to \tau_2}$$

- Third approximate: **use fun rule**

$$\frac{\dfrac{\{f : \delta ; x : \beta\} \mid\!- \text{f (f x)} : \varepsilon}{\{x : \beta\} \mid\!- (\text{fun f -> f (f x)}) : \gamma}}{\{\ \} \mid\!- (\text{fun x -> fun f -> f(f x)}) : \alpha}$$

- $\alpha \equiv (\beta \to \gamma); \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

$$\frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash (e_1\ e_2) : \tau_2}$$

- Fourth approximate: **use app rule**

$$\cfrac{\cfrac{\cfrac{\{f{:}\delta;\ x{:}\beta\} \vdash f : \varphi \rightarrow \varepsilon \quad \{f{:}\delta;\ x{:}\beta\} \vdash f\ x : \varphi}{\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \vdash (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \vdash (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

$$\frac{}{\Gamma \vdash x : \sigma} \quad \text{if } \Gamma(x) = \sigma$$

- Fifth approximate: **use var rule**, get constraint $\delta \equiv \varphi \rightarrow \varepsilon$, Solve with same
- Apply to next sub-proof

$$\cfrac{\cfrac{\cfrac{\{f{:}\delta;\ x{:}\beta\} \vdash f : \varphi \rightarrow \varepsilon \quad \{f{:}\delta;\ x{:}\beta\} \vdash f\ x : \varphi}{\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \vdash (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \vdash (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\delta \equiv \varphi \rightarrow \varepsilon\}$

$$\cfrac{\cfrac{\cfrac{\ldots \quad \{f{:}\varphi \rightarrow \varepsilon;\ x{:}\beta\} \vdash f\ x : \varphi}{\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \vdash (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \vdash (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

$$\frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_2 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash (e_1\ e_2) : \tau_2}$$

- Current subst: $\{\delta \equiv \varphi \rightarrow \varepsilon\}$ Use **App Rule**

$$\cfrac{\cfrac{\cfrac{\ldots \quad \cfrac{\{f{:}\varphi\rightarrow\varepsilon;\ x{:}\beta\} \vdash f{:}\zeta\rightarrow\varphi \quad \{f{:}\varphi\rightarrow\varepsilon;\ x{:}\beta\} \vdash x{:}\zeta}{\{f{:}\varphi \rightarrow \varepsilon;\ x{:}\beta\} \vdash f\ x : \varphi}}{\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \vdash (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \vdash (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\delta \equiv \varphi \rightarrow \varepsilon\}$
- **Var rule**: Solve $\zeta \rightarrow \varphi \equiv \varphi \rightarrow \varepsilon$ **Unification**

$$\cfrac{\cfrac{\cfrac{\ldots \quad \cfrac{\cfrac{}{\{f{:}\varphi\rightarrow\varepsilon;\ x{:}\beta\} \vdash f{:}\zeta\rightarrow\varphi} \quad \{f{:}\varphi\rightarrow\varepsilon;\ x{:}\beta\} \vdash x{:}\zeta}{\{f{:}\varphi \rightarrow \varepsilon;\ x{:}\beta\} \vdash f\ x : \varphi}}{\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \vdash (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \vdash (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\zeta \equiv \varepsilon,\ \varphi \equiv \varepsilon\} \circ \{\delta \equiv \varphi \rightarrow \varepsilon\}$
- **Var rule**: Solve $\zeta \rightarrow \varphi \equiv \varphi \rightarrow \varepsilon$ **Unification**

$$\cfrac{\cfrac{\cfrac{\ldots \quad \cfrac{\cfrac{}{\{f{:}\varphi\rightarrow\varepsilon;\ x{:}\beta\} \vdash f{:}\zeta\rightarrow\varphi} \quad \{f{:}\varphi\rightarrow\varepsilon;\ x{:}\beta\} \vdash x{:}\zeta}{\{f{:}\varphi \rightarrow \varepsilon;\ x{:}\beta\} \vdash f\ x : \varphi}}{\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \vdash (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\ \} \vdash (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \rightarrow \gamma);\ \gamma \equiv (\delta \rightarrow \varepsilon)$

Type Inference - Example

- Current subst: $\{\zeta\equiv\varepsilon,\ \varphi\equiv\varepsilon,\ \delta\equiv\varepsilon\to\varepsilon\}$
- **Apply to next sub-proof**

$$\begin{array}{c}
\text{(done)}\ldots \qquad \{f:\varphi\to\varepsilon;\ x:\beta\} \vdash x:\zeta \\ \hline
\ldots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \vdash f\ x : \varphi \\ \hline
\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon \\ \hline
\{x : \beta\} \vdash (\text{fun } f \to f\ (f\ x)) : \gamma \\ \hline
\{\ \} \vdash (\text{fun } x \to \text{fun } f \to f\ (f\ x)) : \alpha
\end{array}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\zeta\equiv\varepsilon,\ \varphi\equiv\varepsilon,\ \delta\equiv\varepsilon\to\varepsilon\}$
- **Apply to next sub-proof**

$$\begin{array}{c}
\text{(done)}\ldots \qquad \{f:\varepsilon\to\varepsilon;\ x:\beta\} \vdash x:\varepsilon \\ \hline
\ldots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \vdash f\ x : \varphi \\ \hline
\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon \\ \hline
\{x : \beta\} \vdash (\text{fun } f \to f\ (f\ x)) : \gamma \\ \hline
\{\ \} \vdash (\text{fun } x \to \text{fun } f \to f\ (f\ x)) : \alpha
\end{array}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\zeta\equiv\varepsilon,\ \varphi\equiv\varepsilon,\ \delta\equiv\varepsilon\to\varepsilon\}$
- **Var rule**: $\varepsilon\equiv\beta$

$$\begin{array}{c}
\hline
\ldots \qquad \{f:\varepsilon\to\varepsilon;\ x:\beta\} \vdash x:\varepsilon \\ \hline
\ldots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \vdash f\ x : \varphi \\ \hline
\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon \\ \hline
\{x : \beta\} \vdash (\text{fun } f \to f\ (f\ x)) : \gamma \\ \hline
\{\ \} \vdash (\text{fun } x \to \text{fun } f \to f\ (f\ x)) : \alpha
\end{array}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\varepsilon\equiv\beta\}\circ\{\zeta\equiv\varepsilon,\ \varphi\equiv\varepsilon,\ \delta\equiv\varepsilon\to\varepsilon\}$
- Solves subproof; return one layer

$$\begin{array}{c}
\hline
\ldots \qquad \{f:\varepsilon\to\varepsilon;\ x:\beta\} \vdash x:\varepsilon \\ \hline
\ldots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \vdash f\ x : \varphi \\ \hline
\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon \\ \hline
\{x : \beta\} \vdash (\text{fun } f \to f\ (f\ x)) : \gamma \\ \hline
\{\ \} \vdash (\text{fun } x \to \text{fun } f \to f\ (f\ x)) : \alpha
\end{array}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\varepsilon\equiv\beta,\ \zeta\equiv\beta,\ \varphi\equiv\beta,\ \delta\equiv\beta\to\beta\}$
- Solves this subproof; return one layer

$$\begin{array}{c}
\ldots \\ \hline
\ldots \quad \{f:\varphi \to \varepsilon;\ x:\beta\} \vdash f\ x : \varphi \\ \hline
\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon \\ \hline
\{x : \beta\} \vdash (\text{fun } f \to f\ (f\ x)) : \gamma \\ \hline
\{\ \} \vdash (\text{fun } x \to \text{fun } f \to f\ (f\ x)) : \alpha
\end{array}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst: $\{\varepsilon\equiv\beta,\ \zeta\equiv\beta,\ \varphi\equiv\beta,\ \delta\equiv\beta\to\beta\}$
- Need to satisfy constraint $\gamma \equiv (\delta \to \varepsilon)$, given subst, becomes: $\gamma \equiv ((\beta\to\beta) \to \beta)$

$$\begin{array}{c}
\ldots \\ \hline
\{f : \delta\ ;\ x : \beta\} \vdash (f\ (f\ x)) : \varepsilon \\ \hline
\{x : \beta\} \vdash (\text{fun } f \to f\ (f\ x)) : \gamma \\ \hline
\{\ \} \vdash (\text{fun } x \to \text{fun } f \to f\ (f\ x)) : \alpha
\end{array}$$

- $\alpha \equiv (\beta \to \gamma);\ \gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst:

$\{\gamma \equiv ((\beta \to \beta) \to \beta), \varepsilon \equiv \beta, \zeta \equiv \beta, \varphi \equiv \beta, \delta \equiv \beta \to \beta\}$

- Solves subproof; return one layer

$$\dfrac{\dfrac{\dfrac{\dots}{\{f : \delta \,;\, x : \beta\} \text{ |- } (f\ (f\ x)) : \varepsilon}}{\{x : \beta\} \text{ |- } (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\,\} \text{ |- } (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \to \gamma)$; $\gamma \equiv (\delta \to \varepsilon)$

Type Inference - Example

- Current subst:

$\{\gamma \equiv ((\beta \to \beta) \to \beta), \varepsilon \equiv \beta, \zeta \equiv \beta, \varphi \equiv \beta, \delta \equiv \beta \to \beta\}$

- Need to satisfy constraint $\alpha \equiv (\beta \to \gamma)$ given subst: $\alpha \equiv (\beta \to ((\beta \to \beta) \to \beta))$

$$\dfrac{\dfrac{\dots}{\{x : \beta\} \text{ |- } (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\,\} \text{ |- } (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

- $\alpha \equiv (\beta \to \gamma)$;

Type Inference - Example

- Current subst:

$\{\alpha \equiv (\beta \to ((\beta \to \beta) \to \beta)),$
$\gamma \equiv ((\beta \to \beta) \to \beta), \varepsilon \equiv \beta, \zeta \equiv \beta, \varphi \equiv \beta, \delta \equiv \beta \to \beta\}$

- Solves subproof; return on layer

$$\dfrac{\dfrac{}{\{x : \beta\} \text{ |- } (\text{fun } f \text{ -> } f\ (f\ x)) : \gamma}}{\{\,\} \text{ |- } (\text{fun } x \text{ -> fun } f \text{ -> } f\ (f\ x)) : \alpha}$$

Type Inference - Example

- Current subst:

$\{\alpha \equiv (\beta \to ((\beta \to \beta) \to \beta)),$
$\gamma \equiv ((\beta \to \beta) \to \beta), \varepsilon \equiv \beta, \zeta \equiv \beta, \varphi \equiv \beta, \delta \equiv \beta \to \beta\}$

- Done: $\boldsymbol{\alpha \equiv (\beta \to ((\beta \to \beta) \to \beta))}$

{ } |- (fun x -> fun f -> f (f x)) : $\boldsymbol{\alpha}$

Type Inference Algorithm

Let infer $(\Gamma, e, \tau) = \sigma$

- Γ is a typing environment (giving polymorphic types to expression variables)
- e is an expression
- τ is a type (with type variables),
- σ is a substitution of types for type variables
- **Idea**: σ represents the constraints on type variables necessary for Γ |- $e : \tau$
- Should have $\sigma(\Gamma)$ |- $e : \sigma(\tau)$ valid
  - Slight abuse of notation: $\sigma(\Gamma)$ is substitution σ applied to all terms in the environment $\Gamma = \{x: \tau \dots\}$ (i.e., $\sigma(\Gamma) = \{x: \sigma(\tau) \dots\}$).

Type Inference Algorithm (All in one!)

infer (Γ, exp, τ) =

- Case *exp* of
  - Var v --> return Unify$\{\tau \equiv$ freshInstance$(\Gamma(v))\}$
    - Replace all quantified type vars by fresh ones
  - Const c --> return Unify$\{\tau \equiv$ freshInstance $\varphi\}$ where Γ |- $c : \varphi$ by the constant rules
  - fun x -> e -->
    - Let α, β be fresh variables
    - Let σ = infer $(\{x: \alpha\} + \Gamma, e, \beta)$
    - Return Unify$(\{\sigma(\tau) \equiv \sigma(\alpha \to \beta)\}) \circ \sigma$
  - App $(e_1\ e_2)$ -->
    - Let α be a fresh variable
    - Let σ_1 = infer$(\Gamma, e_1, \alpha \to \tau)$
    - Let σ_2 = infer$(\sigma(\Gamma), e_2, \sigma(\alpha))$
    - Return $\sigma_2 \circ \sigma_1$
  - If e_1 then e_2 else e_3 -->
    - Let σ_1 = infer$(\Gamma, e_1,$ bool$)$
    - Let σ_2 = infer$(\sigma\Gamma, e_2, \sigma_1(\tau))$
    - Let σ_3 = infer$(\sigma_2 \circ \sigma_1(\Gamma), e_2, \sigma_2 \circ \sigma(\tau))$
    - Return $\sigma_3 \circ \sigma_2 \circ \sigma_1$
  - let $x = e_1$ in e_2 -->
    - Let α be a fresh variable
    - Let σ_1 = infer(Γ, e_1, α)
    - Let σ_2 = infer$(\{x{:}\text{GEN}(\sigma_1(\Gamma), \sigma_1(\alpha))\} + \sigma_1(\Gamma), e_2, \sigma_1(\tau))$
    - Return $\sigma_2 \circ \sigma_1$
  - let rec $x = e_1$ in e_2 -->
    - Let α be a fresh variable
    - Let σ_1 = infer$(\{x: \alpha\} + \Gamma, e_1, \alpha)$
    - Let σ_2 = infer$(\{x{:}\text{GEN}(\sigma_1(\Gamma), \sigma_1(\alpha))\} + \sigma_1(\Gamma), e_2, \sigma_1(\tau))$
    - Return $\sigma_2 \circ \sigma_1$

Type Inference Algorithm

infer (Γ, *exp*, τ) =

- Case *exp* of
  - Var *v* --> return Unify{$\tau \equiv$ freshInstance($\Gamma(v)$)}
    - Replace all quantified type vars by fresh ones
  - Const *c* --> return Unify{$\tau \equiv$ freshInstance φ } where Γ |- *c* : φ by the constant rules
  - fun *x* -> *e* -->
    - Let α, β be fresh variables
    - Let σ = infer ({x: α} + Γ, *e*, β)
    - Return Unify({$\sigma(\tau) \equiv \sigma(\alpha \to \beta)$}) $\circ\ \sigma$

Inference Example (Repeat)

- Fifth approximate: **use var rule**, get constraint $\delta \equiv \varphi \to \varepsilon$, Solve with same
- Apply to next sub-proof

$$\frac{}{\{f{:}\delta;\ x{:}\beta\} \text{ |- } f : \varphi \to \varepsilon}$$

infer (Γ, *exp*, τ) =

- Case *exp* of
  - Var *v* --> return Unify{$\tau \equiv$ freshInstance($\Gamma(v)$)}
    - Replace all quantified type vars by fresh ones

Inference Example (Repeat)

- What do we do here?

$$\frac{}{\{f{:}\ \forall\ \delta.\ \delta \to \delta\ ;\ x{:}\beta\} \text{ |- } f : \varphi \to \varepsilon}$$

- And here?

$$\{f{:}\ \forall\ \varepsilon.\ \varepsilon \to \varepsilon\ ;\ x{:}\beta\} \text{ |- } f : \varphi \to \varepsilon$$

infer (Γ, *exp*, τ) =

- Case *exp* of
  - Var *v* --> return Unify{$\tau \equiv$ freshInstance($\Gamma(v)$)}
    - Replace all quantified type vars by fresh ones

Inference Example (Repeat)

$$\frac{\{x : \tau_1\} + \Gamma \text{ |- } e : \tau_2}{\Gamma \text{ |- fun } x \text{ -> } e : \tau_1 \to \tau_2}$$

- Third approximate: **use fun rule**

$$\frac{\dfrac{\dots}{\{x : \beta\} \text{ |- (fun f -> f (f x))} : \gamma}}{\{\ \} \text{ |- (fun x -> fun f -> f(f x))} : \alpha}$$

- $\alpha \equiv (\beta \to \gamma)$;

infer (Γ, *exp*, τ) =

- Case *exp* of
  - fun *x* -> *e* -->
    - Let β, γ be fresh variables
    - Let σ = infer ({x: β} + Γ, *e*, γ)
    - Return Unify({$\sigma(\tau) \equiv \sigma(\beta \to \gamma)$}) $\circ\ \sigma$

Type Inference Algorithm (cont)

- Case *exp* of
  - App (e_1 e_2) -->
    - Let α be a fresh variable
    - Let σ_1 = infer(Γ, e_1, $\alpha \to \tau$)
    - Let σ_2 = infer($\sigma_1(\Gamma)$, e_2, $\sigma_1(\alpha)$)
    - Return $\sigma_2 \circ \sigma_1$

Type Inference - Example

$$\frac{\Gamma \text{ |- } e_1 : \tau_1 \to \tau_2 \quad \Gamma \text{ |- } e_2 : \tau_1}{\Gamma \text{ |- } (e_1\ e_2) : \tau_2}$$

- Fourth approximate: **use app rule**

$$\frac{\{f{:}\delta;\ x{:}\beta\}\text{|- } f : \varphi \to \varepsilon \quad \{f{:}\delta;\ x{:}\beta\}\text{|- } f\ x : \varphi}{\{f : \delta\ ;\ x : \beta\} \text{ |- } (f\ (f\ x)) : \varepsilon}$$

- Case *exp* of
  - App (e_1 e_2) -->
    - Let α be a fresh variable
    - Let σ_1 = infer(Γ, e_1, $\alpha \to \tau$)
    - Let σ_2 = infer($\sigma(\Gamma)$, e_2, $\sigma(\alpha)$)
    - Return $\sigma_2 \circ \sigma_1$

Type Inference Algorithm (cont)

- Case *exp* of
  - If e_1 then e_2 else e_3 -->
    - Let σ_1 = infer(Γ, e_1, bool)
    - Let σ_2 = infer($\sigma_1(\Gamma)$, e_2, $\sigma_1(\tau)$)
    - Let σ_3 = infer($\sigma_2 \circ \sigma_1(\Gamma)$, e_2, $\sigma_2 \circ \sigma(\tau)$)
    - Return $\sigma_3 \circ \sigma_2 \circ \sigma_1$

- If_then_else rule:

$$\frac{\Gamma \mid\!- e_1 : \text{bool} \quad \Gamma \mid\!- e_2 : \tau \quad \Gamma \mid\!- e_3 : \tau}{\Gamma \mid\!- (\text{if } e_1 \text{ then } e_2 \text{ else } e_3) : \tau}$$

Type Inference Algorithm (cont)

- Case *exp* of
  - let $x = e_1$ in e_2 -->
    - Let α be a fresh variable
    - Let σ_1 = infer(Γ, e_1, α)
    - Let σ_2 = infer({x:GEN($\sigma_1(\alpha)$, $\sigma_1(\Gamma)$)} + $\sigma_1(\Gamma)$, e_2, $\sigma_1(\tau)$)
    - Return $\sigma_2 \circ \sigma_1$

- let rule:

$$\frac{\Gamma \mid\!- e_1 : \tau_1 \quad \{x : \text{Gen}(\tau_1, \Gamma)\} + \Gamma \mid\!- e_2 : \tau_2}{\Gamma \mid\!- (\text{let } x = e_1 \text{ in } e_2) : \tau_2}$$

Type Inference Algorithm (cont)

- Case *exp* of
  - let rec $x = e_1$ in e_2 -->
    - Let α be a fresh variable
    - Let σ_1 = infer({x: α} + Γ, e_1, α)
    - Let σ_2 = infer({x:GEN($\sigma_1(\alpha)$, $\sigma_1(\Gamma)$)} + $\sigma_1(\Gamma)$, e_2, $\sigma_1(\tau)$)
    - Return $\sigma_2 \circ \sigma_1$

- let rec rule:

$$\frac{\{x : \tau_1\} + \Gamma \mid\!- e_1 : \tau_1 \quad \{x : \text{Gen}(\tau_1, \Gamma)\} + \Gamma \mid\!- e_2 : \tau_2}{\Gamma \mid\!- (\text{let rec } x = e_1 \text{ in } e_2) : \tau_2}$$

Type Inference Algorithm (cont)

- To infer a type, introduce type_of
- Let α be a fresh variable
- type_of (Γ, e) =
  let α be a fresh variable in
  let σ = infer (Γ, e, α)
  in $\sigma(\alpha)$

- Need substitution!
- Need an algorithm for Unif!

Reminder: Type Terms

- Terms made from constructors and variables
- Reminder:
  - Monomorpic Types (τ):
    - Basic Types: int, bool, float, string, unit, …
    - Type Variables: α, β, γ, δ, ε
    - Compound Types: $\alpha \rightarrow \beta$, int * string, bool list, …
  - Polymorphic Types:
    - Monomorphic types τ
    - Universally quantified monomorphic types $\forall \alpha_1, \ldots, \alpha_n . \tau$
    - Can think of τ as same as $\forall . \tau$

Reminder: Type Terms

- Terms made from constructors and variables
- Constructors may be applied to arguments (other terms) to make new terms
- Variables and constructors with no arguments are base cases
- Constructors applied to different number of arguments (arity) considered different
- Substitution of terms for variables

Substitution Implementation

```
type term = Variable of string
          | Constructor of (string * term list)

let rec subst var_name residue term =
    match term with
        Variable name ->
            if var_name = name
            then residue
            else term
       | Constructor (c, tys) ->
            let newt = List.map (subst var_name residue) tys
            in Constructor (c, newt);;
```


Unification Problem

Given a set of pairs of terms ("equations")

$\{(s_1, t_1), (s_2, t_2), \ldots, (s_n, t_n)\}$ *

(the *unification problem*) does there exist a substitution σ (the *unification solution*) of terms for variables such that

$\sigma(s_i)$ is the same as $\sigma(t_i)$,

for all $i = 1, \ldots, n$?

- **Think of these pairs as {("s_1= t_1"), ("s_2= t_2"), …, ("s_n= t_n")}**
  - This is the notation we're going to use in the example

Uses for Unification

- Type Inference and type checking
- Pattern matching as in OCaml
  - Can use a simplified version of algorithm
- Logic Programming - Prolog
- Simple parsing

Unification Algorithm

- Let $S = \{(s_1= t_1), (s_2= t_2), \ldots, (s_n= t_n)\}$ be a unification problem.
  - Unif(S) returns a substitution
- Case S = { }: Unif(S) = Identity function
  - (i.e., no substitution)
- Case $S = \{(s = t)\} \cup S'$: Four main steps
  - Delete, Decompose, Orient, Eliminate

Unification Algorithm for $S = \{(s = t)\} \cup S'$

- Delete: if s is t (s and t are the same term) then
  Unif(S) = Unif(S')
- Decompose: if s is $f(q_1, \ldots, q_m)$ and t is $f(r_1, \ldots, r_m)$ (**same f, same m!**), then
  $\mathrm{Unif}(S) = \mathrm{Unif}(\{(q_1= r_1), \ldots, (q_m=r_m)\} \cup S')$
- Orient: if t is x (a variable), and s is not a variable,
  $\mathrm{Unif}(S) = \mathrm{Unif}(\{(x = s)\} \cup S')$

Unification Algorithm for $S = \{(s = t)\} \cup S'$

- Eliminate: if s is x (a variable), and x does not occur in t (**use "occurs (x, t)" check!**) then
  - Let $\varphi = \{x \to t\}$
  - Let $\psi = \mathrm{Unif}(\varphi(S'))$
  - $\mathrm{Unif}(S) = \{x \to \psi(t)\} \circ \psi$
  - Be careful when composing substitutions:
  - $\{x \to a\} \circ \{y \to b\} =$
    $\{y \to (\{x \to a\}(b))\} \circ \{x \to a\}$ if y not in a

Tricks for Efficient Unification

- Don't return substitution, rather do it incrementally
- Make substitution be constant time
  - Requires implementation of terms to use mutable structures (or possibly lazy structures)
  - We won't discuss these

Example

- x,y,z variables, f,g constructors
- Unify {(f(x) = f(g(f(z),y))), (g(y,y) = x)} = ?

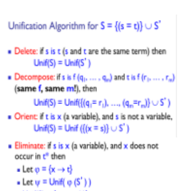
Example

- x,y,z variables, f,g constructors
- Unify {(f(x) = f(g(f(z),y))), (g(y,y) = x)} = ?
- For example:
  - X list = (Z list * Y) list
  - (Y * Y) = X

Example

- x,y,z variables, f,g constructors
- S = {(f(x) = f(g(f(z),y))), (g(y,y) = x)} is nonempty
- Unify {(f(x) = f(g(f(z),y))), (g(y,y) = x)} = ?

Example


Unification Algorithm for S = {(s = t)} ∪ S'

- Delete: if s is t (s and t are the same term) then
  Unif(S) = Unif(S')
- Decompose: if s is f (q_1, … , q_m) and t is f (r_1, … , r_m) (same f, same m!), then
  Unif(S) = Unif({(q_1= r_1), …, (q_m=r_m)} ∪ S')
- Orient: if t is x (a variable), and s is not a variable,
  Unif(S) = Unif ({(x = s)} ∪ S')
- Eliminate: if s is x (a variable), and x does not occur in t* then
  - Let φ = {x → t}
  - Let ψ = Unif(φ (S'))
  - Unif(S) = {x → ψ(t)} o ψ

- x,y,z variables, f,g constructors
- Pick a pair: (g(y,y) = x)
- Unify {(f(x) = f(g(f(z),y))), (g(y,y) = x)} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (g(y,y) = x)
- Orient: (x = g(y,y))
- Unify {(f(x) = f(g(f(z),y))), (g(y,y) = x)} =
  Unify {(f(x) = f(g(f(z),y))), (x = g(y,y))}

by Orient

Example

- x,y,z variables, f,g constructors

- Unify {(f(x) = f(g(f(z),y))), (x = g(y,y))} = ?

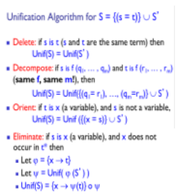
Example

- x,y,z variables, f,g constructors
- {(f(x) = f(g(f(z),y))), (x = g(y,y))} is non-empty

- Unify {(f(x) = f(g(f(z),y))), (x = g(y,y))} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (x = g(y,y))



Unification Algorithm for S = {(s = t)} ∪ S'

- Delete: if s is t (s and t are the same term) then
  Unif(S) = Unif(S')
- Decompose: if s is f (q_1, … , q_m) and t is f (r_1, … , r_m) (same f, same m!), then
  Unif(S) = Unif({(q_1= r_1), …, (q_m=r_m)} ∪ S')
- Orient: if t is x (a variable), and s is not a variable,
  Unif(S) = Unif ({(x = s)} ∪ S')
- Eliminate: if s is x (a variable), and x does not occur in t* then
  - Let φ = {x → t}
  - Let ψ = Unif(φ (S'))
  - Unif(S) = {x → ψ(t)} o ψ

- Unify {(f(x) = f(g(f(z),y))), (x = g(y,y))} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (x = g(y,y))
- Eliminate x with substitution {x→ g(y,y)}
  - Check: x not in g(y,y)
- Unify {(f(x) = f(g(f(z),y))), (x = g(y,y))} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (x = g(y,y))
- Eliminate x with substitution {x→ g(y,y)}

- Unify {(f(x) = f(g(f(z),y))), (x = g(y,y))} =
  Unify {(f(g(y,y)) = f(g(f(z),y)))}
  o {x→ g(y,y)}

Example

- x,y,z variables, f,g constructors

- Unify {(f(g(y,y)) = f(g(f(z),y)))}
  o {x→ g(y,y)} = ?

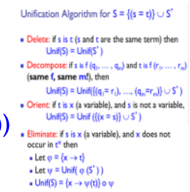
Example

- x,y,z variables, f,g constructors
- {(f(g(y,y)) = f(g(f(z),y)))} is non-empty

- Unify {(f(g(y,y)) = f(g(f(z),y)))}
  o {x→ g(y,y)} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (f(g(y,y)) = f(g(f(z),y)))


Unification Algorithm for S = {(s = t)} ∪ S'

- Delete: if s is t (s and t are the same term) then
  Unif(S) = Unif(S')
- Decompose: if s is f (q_1, … , q_m) and t is f (r_1, … , r_m) (same f, same m!), then
  Unif(S) = Unif({(q_1= r_1), …, (q_m=r_m)} ∪ S')
- Orient: if t is x (a variable), and s is not a variable,
  Unif(S) = Unif ({(x = s)} ∪ S')
- Eliminate: if s is x (a variable), and x does not occur in t* then
  - Let φ = {x → t}
  - Let ψ = Unif(φ (S'))
  - Unif(S) = {x → ψ(t)} o ψ

- Unify {(f(g(y,y)) = f(g(f(z),y)))}
  o {x→ g(y,y)} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (f(g(y,y)) = f(g(f(z),y)))
- Decompose:(f(g(y,y)) = f(g(f(z),y))) becomes {(g(y,y) = g(f(z),y))}

- Unify {(f(g(y,y)) = f(g(f(z),y)))}
  o {x→ g(y,y)} =
  Unify {(g(y,y) = g(f(z),y))} o {x→ g(y,y)}

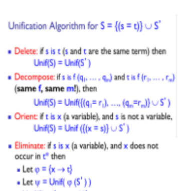
Example

- x,y,z variables, f,g constructors
- {(g(y,y) = g(f(z),y))} is non-empty

- Unify {(g(y,y) = g(f(z),y))}
  o {x→ g(y,y)} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (g(y,y) = g(f(z),y))


Unification Algorithm for S = {(s = t)} ∪ S'

- Delete: if s is t (s and t are the same term) then
  Unif(S) = Unif(S')
- Decompose: if s is f (q_1, … , q_m) and t is f (r_1, … , r_m) (same f, same m!), then
  Unif(S) = Unif({(q_1= r_1), …, (q_m=r_m)} ∪ S')
- Orient: if t is x (a variable), and s is not a variable,
  Unif(S) = Unif ({(x = s)} ∪ S')
- Eliminate: if s is x (a variable), and x does not occur in t* then
  - Let φ = {x → t}
  - Let ψ = Unif(φ (S'))
  - Unif(S) = {x → ψ(t)} o ψ

- Unify {(g(y,y) = g(f(z),y))}
  o {x→ g(y,y)} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (f(g(y,y)) = f(g(f(z),y)))
- Decompose: (g(y,y)) = g(f(z),y)) becomes {(y = f(z)); (y = y)}

- Unify {(g(y,y) = g(f(z),y))} o {x→ g(y,y)} =
  Unify {(y = f(z)); (y = y)} o {x→ g(y,y)}

Example

- x,y,z variables, f,g constructors
- Unify {(y = f(z)); (y = y)} o {x→ g(y,y)} = ?

Example

- x,y,z variables, f,g constructors
- {(y = f(z)); (y = y)} o {x→ g(y,y)} is non-empty
- Unify {(y = f(z)); (y = y)} o {x→ g(y,y)} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (y = f(z))
- Unify {(y = f(z)); (y = y)} o {x→ g(y,y)} = ?

Unification Algorithm for S = {(s = t)} ∪ S'

- Delete: if s is t (s and t are the same term) then
  Unif(S) = Unif(S')
- Decompose: if s is f (q_1, … , q_m) and t is f (r_1, … , r_m) (same f, same m!), then
  Unif(S) = Unif({(q_1= r_1), …, (q_m=r_m)} ∪ S')
- Orient: if t is x (a variable), and s is not a variable,
  Unif(S) = Unif ({(x = s)} ∪ S')
- Eliminate: if s is x (a variable), and x does not occur in t* then
  - Let φ = {x → t}
  - Let ψ = Unif(φ (S'))
  - Unif(S) = {x → ψ(t)} o ψ

Example

- x,y,z variables, f,g constructors
- Pick a pair: (y = f(z))
- Eliminate y with {y → f(z)}
- Unify {(y = f(z)); (y = y)} o {x→ g(y,y)} = Unify {(f(z) = f(z))}
  o {y → f(z)} o {x→ g(y,y)}=
  Unify {(f(z) = f(z))}
  o {y → f(z); x→ g(f(z), f(z))}

- Eliminate: if s is x (a variable), and x does not occur in t* then
  - Let φ = {x → t}
  - Let ψ = Unif(φ (S'))
  - Unif(S) = {x → ψ(t)} o ψ

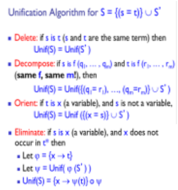
Example

- x,y,z variables, f,g constructors
- Unify {(f(z) = f(z))}
  o {y → f(z); x→ g(f(z), f(z))} = ?

Example

- x,y,z variables, f,g constructors
- {(f(z) = f(z))} is non-empty
- Unify {(f(z) = f(z))}
  o {y → f(z); x→ g(f(z), f(z))} = ?

Example



Unification Algorithm for $S = \{(s = t)\} \cup S'$

- Delete: if s is t (s and t are the same term) then
  Unif(S) = Unif(S')
- Decompose: if s is f ($q_1, \ldots, q_m$) and t is f ($r_1, \ldots, r_m$) (same f, same m!), then
  Unif(S) = Unif($\{(q_1 = r_1), \ldots, (q_m = r_m)\} \cup S'$)
- Orient: if t is x (a variable), and s is not a variable,
  Unif(S) = Unif ($\{(x = s)\} \cup S'$)
- Eliminate: if s is x (a variable), and x does not occur in t* then
  - Let $\varphi = \{x \to t\}$
  - Let ψ = Unif(φ (S'))
  - Unif(S) = $\{x \to \psi(t)\} \circ \psi$

- x,y,z variables, f,g constructors
- Pick a pair: (f(z) = f(z))
- Unify {(f(z) = f(z))}
  o {y → f(z); x→ g(f(z), f(z))} = ?

Example

- x,y,z variables, f,g constructors
- Pick a pair: (f(z) = f(z))
- Delete
- Unify {(f(z) = f(z))}
  o {y → f(z); x→ g(f(z), f(z))} =
  Unify {} o {y → f(z); x→ g(f(z), f(z))}

Example

- x,y,z variables, f,g constructors
- Unify {} o {y → f(z); x→ g(f(z), f(z))} = ?

Example

- x,y,z variables, f,g constructors
- {} is empty
- Unify {} = identity function
- Unify {} o {y → f(z); x→ g(f(z), f(z))} =
  {y → f(z); x→ g(f(z), f(z))}

Example

- Unify {(f(x) = f(g(f(z),y))), (g(y,y) = x)} =
  {y → f(z); x→ g(f(z), f(z))}

f(x) = f(g(f(z), y))
→ f(g(f(z), f(z))) = f(g(f(z), f(z)))

g(y , y) = x
→ g(f(z),f(z)) = g(f(z), f(z))

Example

- Unify {(f(x) = f(g(f(z),y))), (g(y,y) = x)} =
  {y → f(z); x→ g(f(z), f(z))}

y → int list, x→ (int list * int list)

f(x) = f(g(f(z), y))
→ f(g(f(z), f(z))) = f(g(f(z), f(z)))

(int list * int list) list = (int list * int list) list

g(y , y) = x
→ g(f(z),f(z)) = g(f(z), f(z))

(int list * int list) = (int list * int list)

Example of Failure: Decompose

- Unify{(f(x,g(y)) = f(h(y),x))}
  Decompose: (f(x,g(y)) = f(h(y),x))
- = Unify {(x = h(y)), (g(y) = x)}
  Orient: (g(y) = x)
- = Unify {(x = h(y)), (x = g(y))}
  Eliminate: (x = h(y))
- = Unify {(h(y) = g(y))} o {x → h(y)}
  - **No rule to apply! Decompose fails!**

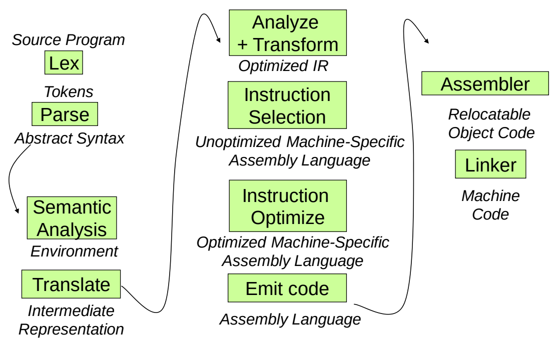
Example of Failure: Occurs Check

- Unify{(f(x,g(x)) = f(h(x),x))}
  Decompose: (f(x,g(x)) = f(h(x),x))
- = Unify {(x = h(x)), (g(x) = x)}
  Orient: (g(y) = x)
- = Unify {(x = h(x)), (x = g(x))}
  - **No rules apply.**

Course Objectives

- New programming paradigm
  - Functional programming
  - Environments and Closures
  - Patterns of Recursion
  - Continuation Passing Style
- Phases of an interpreter / compiler
  - Lexing and parsing
  - Type systems
  - Interpretation
- Programming Language Semantics
  - Lambda Calculus
  - Operational Semantics
  - Axiomatic Semantics

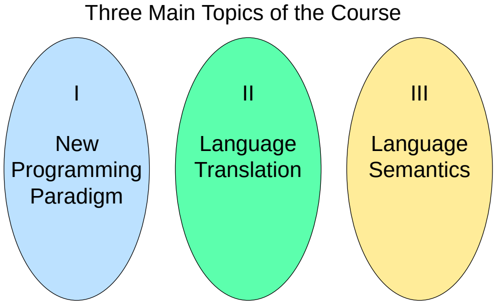
Major Phases of a Compiler



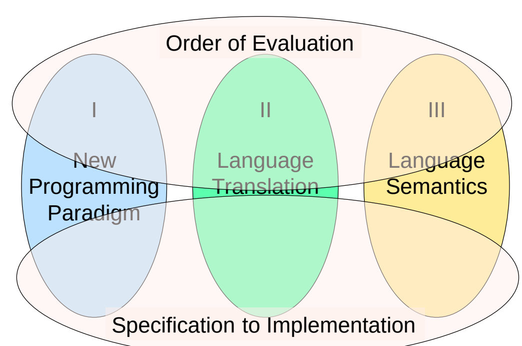
Modified from "Modern Compiler Implementation in ML", by Andrew Appel

Programming Languages & Compilers

Three Main Topics of the Course

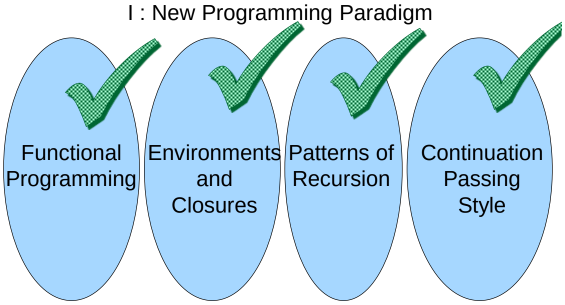


Programming Languages & Compilers



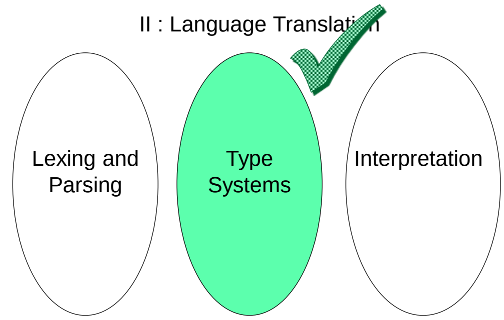
Programming Languages & Compilers

I : New Programming Paradigm



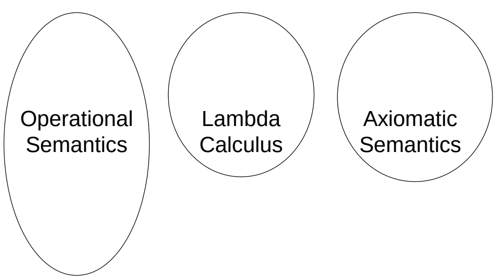
Programming Languages & Compilers

II : Language Translation



Programming Languages & Compilers

III : Language Semantics



Meta-discourse

- Language Syntax and Semantics
- Syntax
  - Regular Expressions, DFSAs and NDFSAs
  - Grammars
- Semantics
  - Natural Semantics
  - Transition Semantics

Language Syntax

- Syntax is the description of which strings of symbols are meaningful expressions in a language
- It takes more than syntax to understand a language; need meaning (semantics) too
- Syntax is the entry point

Syntax of English Language

- Pattern 1

Subject	Verb
David	*sings*
The dog	*barked*
Susan	*yawned*

- Pattern 2

Subject	Verb	Direct Object
David	*sings*	*ballads*
The professor	*wants*	*to retire*
The jury	*found*	*the defendant guilty*

Elements of Syntax

- Character set – previously always ASCII, now often 64 character sets
- Keywords – usually reserved
- Special constants – cannot be assigned to
- Identifiers – can be assigned to
- Operator symbols
- Delimiters (parenthesis, braces, brackets)
- Blanks (aka white space)

Elements of Syntax

- Expressions
  - if ... then begin ... ; ... end else begin ... ; ... end
- Type expressions
  - $typexpr_1$ -> $typexpr_2$
- Declarations (in functional languages)
  - let $pattern_1$ = $expr_1$ in *expr*
- Statements (in imperative languages)
  - a = b + c
- Subprograms
  - let $pattern_1$ = let rec inner = … in *expr*

Elements of Syntax

- Modules
- Interfaces
- Classes (for object-oriented languages)

Lexing and Parsing

- Converting strings to abstract syntax trees done in two phases
  - **Lexing:** Converting string (or streams of characters) into lists (or streams) of tokens (the "words" of the language)
    - Specification Technique: Regular Expressions
  - **Parsing:** Convert a list of tokens into an abstract syntax tree
    - Specification Technique: BNF Grammars

Formal Language Descriptions

- Regular expressions, regular grammars, finite state automata
- Context-free grammars, BNF grammars, syntax diagrams
- Whole family more of grammars and automata – covered in automata theory

Grammars

- Grammars are formal descriptions of which strings over a given character set are in a particular language
- Language designers write grammar
- Language implementers use grammar to know what programs to accept
- Language users use grammar to know how to write legitimate programs