Programming Languages and Compilers (CS 421)

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Based in part on slides by Mattox Beckman, as updated by Vikram Adve and Gul Agha

Axiomatic Semantics

- Also called Floyd-Hoare Logic
- Based on formal logic (first order predicate calculus)
- Axiomatic Semantics is a logical system built from *axioms* and *inference rules*
- Mainly suited to simple imperative programming languages

Axiomatic Semantics

- Used to formally prove a property (*post-condition*) of the *state* (the values of the program variables) after the execution of program, assuming another property (*pre-condition*) of the state holds before execution

Axiomatic Semantics

- Goal: Derive statements of form

  $$\{P\}\ C\ \{Q\}$$

  - P , Q logical statements about state, P precondition, Q postcondition, C program

- Example: $\{x = 1\}\ x := x + 1\ \{x = 2\}$

Axiomatic Semantics

- *Approach:* For each type of language statement, give an axiom or inference rule stating how to derive assertions of form

$$\{P\}\ C\ \{Q\}$$

where C is a statement of that type

- Compose axioms and inference rules to build proofs for complex programs

Axiomatic Semantics

- An expression {P} C {Q} is a *partial correctness* statement
- For *total correctness* must also prove that C terminates (i.e. doesn't run forever)
  - Written: [P] C [Q]
- Will only consider partial correctness here

Language

- We will give rules for simple imperative language

<command>
::= <variable> := <term>
| <command>; … ;<command>
| if <statement> then <command> else <command> fi
| while <statement> do <command> od

- Could add more features, like for-loops

Substitution

- Notation: P[e/v] (sometimes P[v <- e])
- Meaning: Replace every v in P by e
- Example:

$$(x + 2)\,[y-1/x] = ((y - 1) + 2)$$

The Assignment Rule

$$\frac{}{\{P\,[e/x]\}\ x := e\ \{P\}}$$

Example:

$$\frac{}{\{\quad ? \quad\}\ x := y\ \{x = 2\}}$$

The Assignment Rule

$$\frac{}{\{P\,[e/x]\,\}\ x := e\ \{P\}}$$

Example:

$$\frac{}{\{\boxed{\_}= 2\ \}\ x := y\ \{\boxed{x} = 2\}}$$

The Assignment Rule

$$\frac{}{\{P\,[e/x]\}\ x := e\ \{P\}}$$

Example:

$$\frac{}{\{\boxed{y} = 2\}\ x := y\ \{\boxed{x} = 2\}}$$

The Assignment Rule

$$\frac{}{\{P\,[e/x]\,\}\ x := e\ \{P\}}$$

Examples:

$$\frac{}{\{y = 2\}\ x := y\ \{x = 2\}}$$

$$\frac{}{\{y = 2\}\ x := 2\ \{y = x\}}$$

$$\frac{}{\{x + 1 = n + 1\}\ x := x + 1\ \ \{x = n + 1\}}$$

$$\frac{}{\{2 = 2\}\ x := 2\ \{x = 2\}}$$

The Assignment Rule – Your Turn

- What is the weakest precondition of

$$x := x + y \ \{x + y = w - x\}?$$

$$\{ \quad ? \quad \}$$

$$x := x + y$$

$$\{x + y = w - x\}$$

The Assignment Rule – Your Turn

- What is the weakest precondition of

$$x := x + y \ \{x + y = w - x\}?$$

$$\{(x + y) + y = w - (x + y)\}$$
$$x := x + y$$
$$\{x + y = w - x\}$$

Precondition Strengthening

$$\frac{P \rightarrow P' \qquad \{P'\}\ C\ \{Q\}}{\{P\}\ C\ \{Q\}}$$

- Meaning: If we can show that P implies P' ($P \rightarrow P'$) and we can show that {P'} C {Q}, then we know that {P} C {Q}
- P is *stronger* than P' means $P \rightarrow P'$

Precondition Strengthening

- Examples:

$$\frac{x = 3 \Rightarrow x < 7 \quad \{x < 7\}\ x := x + 3\ \{x < 10\}}{\{x = 3\}\ x := x + 3\ \{x < 10\}}$$

$$\frac{\text{True} \Rightarrow 2 = 2 \quad \{2 = 2\}\ x := 2\ \{x = 2\}}{\{\text{True}\}\ x := 2\ \{x = 2\}}$$

$$\frac{x=n \Rightarrow x+1=n+1 \quad \{x+1=n+1\}\ x:=x+1\ \{x=n+1\}}{\{x=n\}\ x:=x+1\ \{x=n+1\}}$$

Which Inferences Are Correct?

$$\frac{\{x > 0 \ \&\ x < 5\}\ x := x * x\ \{x < 25\}}{\{x = 3\}\ x := x * x\ \{x < 25\}}$$

$$\frac{\{x = 3\}\ x := x * x\ \{x < 25\}}{\{x > 0 \ \&\ x < 5\}\ x := x * x\ \{x < 25\}}$$

$$\frac{\{x * x < 25\}\ x := x * x\ \{x < 25\}}{\{x > 0 \ \&\ x < 5\}\ x := x * x\ \{x < 25\}}$$

Which Inferences Are Correct?

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$$\cancel{\frac{\{x = 3\}\ x := x * x\ \{x < 25\}}{\{x > 0 \;\&\; x < 5\}\ x := x * x\ \{x < 25\}}}$$

$$\frac{\{x * x < 25\}\ x := x * x\ \{x < 25\}}{\{x > 0 \;\&\; x < 5\}\ x := x * x\ \{x < 25\}}$$

Sequencing

$$\frac{\{P\}\ C_1\ \{Q\} \qquad \{Q\}\ C_2\ \{R\}}{\{P\}\ C_1;\ C_2\ \{R\}}$$

- Example:

$$\frac{\{z = z \;\&\; z = z\}\ x := z\ \{x = z \;\&\; z = z\} \qquad \{x = z \;\&\; z = z\}\ y := z\ \{x = z \;\&\; y = z\}}{\{z = z \;\&\; z = z\}\ x := z;\ y := z\ \{x = z \;\&\; y = z\}}$$

Sequencing

$$\frac{\{P\}\ C_1\ \{Q\} \qquad \{Q\}\ C_2\ \{R\}}{\{P\}\ C_1;\ C_2\ \{R\}}$$

- Example:

$$\frac{\{z = z \;\&\; z = z\}\ x := z\ \{x = z \;\&\; z = z\} \qquad \{x = z \;\&\; z = z\}\ y := z\ \{x = z \;\&\; y = z\}}{\{z = z \;\&\; z = z\}\ x := z;\ y := z\ \{x = z \;\&\; y = z\}}$$

Postcondition Weakening

$$\frac{\{P\}\ C\ \{Q'\} \qquad Q' \rightarrow Q}{\{P\}\ C\ \{Q\}}$$

Example:

$$\frac{\{z = z\ \&\ z = z\}\ x := z;\ y := z\ \{x = z\ \&\ y = z\} \qquad (x = z\ \&\ y = z) \rightarrow (x = y)}{\{z = z\ \&\ z = z\}\ x := z;\ y := z\ \{x = y\}}$$

Rule of Consequence

$$\frac{P \rightarrow P' \qquad \{P'\}\ C\ \{Q'\} \qquad Q' \rightarrow Q}{\{P\}\ C\ \{Q\}}$$

- Logically equivalent to the combination of Precondition Strengthening and Postcondition Weakening
- Uses $P \rightarrow P'$ and $Q' \rightarrow Q$

If Then Else

$$\frac{\{P \text{ and } B\}\ C_1\ \{Q\} \quad \{P \text{ and } (\text{not } B)\}\ C_2\ \{Q\}}{\{P\}\ \textbf{if}\ B\ \textbf{then}\ C_1\ \textbf{else}\ C_2\ \textbf{fi}\ \{Q\}}$$

- Example: Want

{y=a}

if x < 0 then y:= y-x else y:= y+x fi

{y=a+|x|}

Suffices to show:

(1) {y=a&x<0} y:=y-x {y=a+|x|} and

(4) {y=a&not(x<0)} y:=y+x {y=a+|x|}

$\{y=a\&x<0\}\ y:=y-x\ \{y=a+|x|\}$

$$\frac{(3)\quad (y=a\&x<0) \rightarrow y-x=a+|x| \qquad (2)\quad \{y-x=a+|x|\}\ y:=y-x\ \{y=a+|x|\}}{(1)\quad \{y=a\&x<0\}\ y:=y-x\ \{y=a+|x|\}}$$

(1) Reduces to (2) and (3) by Precondition Strengthening

(2) Follows from assignment axiom

(3) Because $x<0 \rightarrow |x| = -x$

$\{y=a\&not(x<0)\}\ y:=y+x\ \{y=a+|x|\}$

(6) $(y=a\&not(x<0)) \rightarrow (y+x=a+|x|)$

(5) $\{y+x=a+|x|\}\ \ y:=y+x\ \ \{y=a+|x\}\}$

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(4) $\{y=a\&not(x<0)\}\ \ y:=y+x\ \ \{y=a+|x|\}$

(4) Reduces to (5) and (6) by Precondition Strengthening

(5) Follows from assignment axiom

(6) Because $not(x<0) \rightarrow |x| = x$

If then else

$$\frac{\begin{array}{ll}(1) & \{y=a \,\&\, x<0\}\, y:=y-x\, \{y=a+|x|\} \\ (4) & \{y=a \,\&\, \text{not}(x<0)\}\, y:=y+x\, \{y=a+|x|\}\end{array}}{\begin{array}{c}\{y=a\} \\ \text{if } x < 0 \text{ then } y:= y-x \text{ else } y:= y+x \\ \{y=a+|x|\}\end{array}}$$

By the if_then_else rule

While

- We need a rule to be able to make assertions about **while** loops.
  - Inference rule because we can only draw conclusions if we know something about the body
  - Let's start with:

$$\frac{\{\ ?\ \}\ C\ \{\ ?\ \}}{\{\ ?\ \}\ \textbf{while}\ B\ \textbf{do}\ C\ \textbf{od}\ \{\ P\ \}}$$

While

- The loop may never be executed, so if we want P to hold after, it had better hold before, so let's try:

$$\frac{\{\ ?\ \}\ C\ \{\ ?\ \}}{\{\ P\ \}\ \textbf{while}\ B\ \textbf{do}\ C\ \textbf{od}\ \{\ P\ \}}$$

While

- If all we know is P when we enter the **while** loop, then we all we know when we enter the body is (P and B)
- If we need to know P when we finish the **while** loop, we had better know it when we finish the loop body:

$$\frac{\{\,P \text{ and } B\,\}\ C\ \{\,P\,\}}{\{\,P\,\}\ \textbf{while}\ B\ \textbf{do}\ C\ \textbf{od}\ \{\,P\,\}}$$

While

- We can strengthen the previous rule because we also know that when the loop is finished, **not B** also holds
- Final **while** rule:

$$\frac{\{\ P \text{ and } B\ \}\ \ C\ \ \{\ P\ \}}{\{\ P\ \}\ \textbf{while}\ \ B\ \ \textbf{do}\ \ C\ \textbf{od}\ \ \{\ P \text{ and not } B\ \}}$$

While

$$\frac{\{\, P \text{ and } B \,\}\ \ C\ \ \{\, P \,\}}{\{\, P \,\}\ \textbf{while}\ \ B\ \ \textbf{do}\ \ C\ \textbf{od}\ \ \{\, P \text{ and not } B \,\}}$$

- P satisfying this rule is called a *loop invariant* because it must hold before and after the each iteration of the loop

While

- **While** rule generally needs to be used together with precondition strengthening and postcondition weakening
- There is NO algorithm for computing the correct P; it requires intuition and an understanding of why the program works

Example

- Let us prove

  {x>= 0 and x = a}

  fact := 1;

  while x > 0 do (fact := fact * x; x := x –1) od

  {fact = a!}

Example

- We need to find a condition P that is true both before and after the loop is executed, and such that

$$(P \text{ and not } x > 0) \rightarrow (fact = a!)$$

Example

- First attempt:

{a! = fact * (x!)}

- Motivation:
- What we want to compute: **a!**
- What we have computed: **fact**

  which is the sequential product of **a** down through **(x + 1)**
- What we still need to compute: **x!**

Example

By post-condition weakening suffices to show

1. {x>=0 and x = a}

   fact := 1;

   while x > 0 do (fact := fact * x; x := x –1) od

   {a! = fact * (x!) and not (x > 0)}

and

2. {a! = fact * (x!) and not (x > 0) } ➔ {fact = a!}

Problem

2. {a! = fact * (x!) and not (x > 0)} ➔ {fact = a!}

- Don't know this if x < 0
- Need to know that x = 0 when loop terminates
- Need a new loop invariant
- Try adding x >= 0
- Then will have x = 0 when loop is done

Example

Second try, combine the two:

P = {a! = fact * (x!) and x >=0}

Again, suffices to show

1. {x>=0 and x = a}
   fact := 1;
   while x > 0 do (fact := fact * x; x := x –1) od
   {P and not x > 0}

and

2. {P and not x > 0} ➔ {fact = a!}

Example

- For 2, we need

  {a! = fact * (x!) and x >=0 and not (x > 0)} ➔ {fact = a!}

  But {x >=0 and not (x > 0)} ➔ {x = 0} so

  fact * (x!) = fact * (0!) = fact

  Therefore

  {a! = fact * (x!) and x >=0 and not (x > 0)} ➔ {fact = a!}

Example

- For 1, by the sequencing rule it suffices to show

3. {x>=0 and x = a}
   fact := 1
   {a! = fact * (x!) and x >=0 }

And

4. {a! = fact * (x!) and x >=0}
   while x > 0 do
   (fact := fact * x; x := x –1) od
   {a! = fact * (x!) and x >=0 and not (x > 0)}

Example

- Suffices to show that

  {a! = fact * (x!) and x >= 0}

  holds before the while loop is entered and that if

  {(a! = fact * (x!)) and x >= 0 and x > 0}

  holds before we execute the body of the loop, then

  {(a! = fact * (x!)) and x >= 0}

  holds after we execute the body

Example

By the assignment rule, we have

{a! = 1 * (x!) and x >= 0}

fact := 1

{a! = fact * (x!) and x >= 0}

Therefore, to show (3), by precondition strengthening, it suffices to show

(x>= 0 and x = a) ➔

(a! = 1 * (x!) and x >= 0)

Example

(x>= 0 and x = a) ➔
(a! = 1 * (x!) and x >= 0)

holds because x = a ➔ x! = a!

Have that {a! = fact * (x!) and x >= 0}
holds at the start of the while loop

Example

To show (4):

{a! = fact * (x!) and x >=0}
while x > 0 do
(fact := fact * x; x := x –1)
od
{a! = fact * (x!) and x >=0 and not (x > 0)}

we need to show that

{(a! = fact * (x!)) and x >= 0}

is a loop invariant

Example

We need to show:

{(a! = fact * (x!)) and x >= 0 and x > 0}

(fact = fact * x; x := x – 1)

{(a! = fact * (x!)) and x >= 0}

We will use assignment rule, sequencing rule and precondition strengthening

Example

By the assignment rule, we have

{(a! = fact * ((x-1)!)) and x – 1 >= 0}

x := x – 1

{(a! = fact * (x!)) and x >= 0}

By the sequencing rule, it suffices to show

{(a! = fact * (x!)) and x >= 0 and x > 0}

fact = fact * x

{(a! = fact * ((x-1)!)) and x – 1 >= 0}

Example

By the assignment rule, we have that

{((a! = (fact * x) * ((x-1)!)) and x – 1 >= 0}

fact = fact * x

{((a! = fact * ((x-1)!)) and x – 1 >= 0}

By Precondition strengthening, it suffices to show that

((a! = fact * (x!)) and x >= 0 and x > 0) ➔ ((a! = (fact * x) * ((x-1)!)) and x – 1 >= 0)

Example

However

$$fact * x * (x - 1)! = fact * x$$

and $(x > 0) \rightarrow x - 1 >= 0$

since x is an integer,so

$$\{(a! = fact * (x!)) \text{ and } x >= 0 \text{ and } x > 0\} \rightarrow$$

$$\{(a! = (fact * x) * ((x-1)!)) \text{ and } x - 1 >= 0\}$$

Example

Therefore, by precondition strengthening

{(a! = fact * (x!)) and x >= 0 and x > 0}

fact = fact * x

{(a! = fact * ((x-1)!)) and x – 1 >= 0}

This finishes the proof