Programming Languages and Compilers (CS 421)

#7: Continuations and Continuation Passing Style (CPS)
#6: Continuation Passing Style transformation, modeling exceptions

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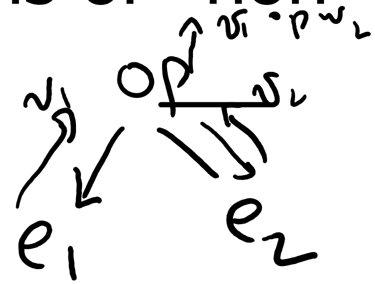
Based on slides by Elsa Gunter, in turn partly based on slides by Mattox Beckman, Vikram Adve and Gul Agha

Continuation Passing Style

- A programming technique for all forms of "non-local" control flow:
  - non-local jumps
  - exceptions
  - general conversion of non-tail calls to tail calls



- Essentially it's a higher-order function version of GOTO

- *Tail-recursion on acid*

Continuations

- Idea: Use functions to represent the control flow of a program

- Method: Each procedure takes a function as an argument to which to pass its result; outer procedure "returns" no result

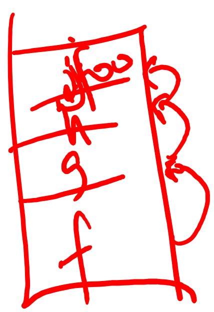
- Function receiving the result called a continuation

- Continuation acts as "accumulator" for work still to be done

Continuation Passing Style

- Writing procedures such that all procedure calls take a continuation to which to give (pass) the result, and return no result, is called continuation passing style (CPS)

- Note: All functions must be in CPS form.



Continuation Passing Style

- A compilation technique to implement non-local control flow, especially useful in interpreters.
- A formalization of non-local control flow in denotational semantics
- Possible intermediate state in compiling functional code

Why CPS?

- Makes order of evaluation explicitly clear
- Allocates variables (to become registers) for each step of computation
- Essentially converts functional programs into imperative ones
  - Major step for compiling to assembly or byte code
- Tail recursion easily identified
- Strict forward recursion converted to tail recursion
  - At the expense of building large closures in heap

Other Uses for Continuations

- CPS designed to preserve order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

Back to recursion and tail-recursion

Compute the product of the numbers in a list:

Not tail recursive:

```
# let rec prod l =
    match l with [] -> 1
    | (x :: rem) -> x * prod rem;;
val prod : int list -> int = <fun>
```

let

Tail recursive:

```
let prod list =
    let rec prod_aux l acc =
        match l with [] -> acc
        | (y :: rest) -> prod_aux rest (acc * y)
    in prod_aux list 1;;
 val prod : int list -> int = <fun>
```

Key idea:

Do work that you have to do after the function call before you call the function, and have an accumulator hold the computed values.

Associativity is crucial.

Doesn't work in general.

Back to recursion and tail-recursion

How do we write the following in a way that is syntactically tail-recursive?

```
# let crazy list = match list with [] -> 0
  | (h::t) -> ((crazy t) + h + 2) * 3

# crazy [4;5;9];;
- : int = 378
```

```
let crazyK list k
= match list with [] -> k 0
| (h::t) -> crazyK t
    (fun r -> k ((r+h+2) * 3))

crazyK list (fun x -> print_int x)
```

Back to recursion and tail-recursion

How do we write the following in a way that is syntactically tail-recursive?

```
# let crazy list = match list with [] -> 0
  | (h::t) -> ((crazy t) + h + 2) * 3
# crazy [4;5;9];;
-   : int = 378

# let rec crazyk list k = match list with [] -> k 0
     | (h::t) -> crazyk t (fun r -> k ((r + h + 2) * 3))
# let justret x = x
# crazyk [4;5;9] justret;;
- : int = 378
```

- Transformed version is tail-recursive, syntactically.
- But not efficient!
- Evaluates the same function the same way…

- The continuation encodes the ``stack"

```
crazyk [4;5] justret
= crazyk [5] (fun r -> justret ((r + 4 + 2) * 3))
= crazyk []  (fun r -> (fun r -> justret ((r + 4 + 2) * 3))
                          ((r + 5 + 2) * 3))
= (fun r -> (fun r -> justret ((r + 4 + 2) * 3))
                          ((r + 5 + 2) * 3))  0
= (fun r -> justret ((r + 4 + 2) * 3)) 21
= 81
```

Now, let’s do this so that we do only *one* small piece of work.

Function can either:

- do some primitive function
- or call another function with a continuation

Example

- Simple reporting continuation:

```
# let report x = (print_int x; print_newline( ) );;
val report : int -> unit = <fun>
```

- Simple function using a continuation:

```
# let addk (a, b) k = k (a + b);;
val addk : int * int -> (int -> 'a) -> 'a = <fun>
# addk (22, 20) report;;
42
- : unit = ()
```

Simple Functions Taking Continuations

- Given a primitive operation, can convert it to pass its result forward to a continuation
- Examples:

```
# let subk (x, y) k = k(x - y);;
val subk : int * int -> (int -> 'a) -> 'a = <fun>
# let eqk (x, y) k = k(x = y);;
val eqk : 'a * 'a -> (bool -> 'b) -> 'b = <fun>
# let timesk (x, y) k = k(x * y);;
val timesk : int * int -> (int -> 'a) -> 'a = <fun>
```

Your turn now

Try Problem 7 on MP2

Try consk

Nesting Continuations

```
# let add_triple (x, y, z) = (x + y) + z;;
val add_triple : int * int * int -> int = <fun>
# let add_triple (x,y,z)=let p = x + y in p + z;;
val add_three : int -> int -> int -> int = <fun>
# let add_triple_k (x, y, z) k =
   addk (x, y) (fun p -> addk (p, z) k );;
val add_triple_k: int * int * int -> (int -> 'a) ->
   'a = <fun>
```

add_three: a different order

- # let add_triple (x, y, z) = x + (y + z);;
- How do we write add_triple_k to use a different order?

add_triple (x, y, z) = let p = y+z in x + p

- let add_triple_k (x, y, z) k =

addk (y, z) (fun p -> addk (x, p) k)

Your turn now

Try Problem 8 on MP4

Conditionals

```
# let not5 x = if (x = 5) then 0 else x
```

let not5 x = let b = (x=5) in
if b then 0 else x

let not5k x k
= eqk (x, 5) (fun b -> if b then k 0
else k x)

Conditionals

```
# let not5 x = if (x = 5) then 0 else x

# let not5k x k =
    eqk (x,5) (fun r -> if r then k 0 else k x)
```

Recursive Functions

- Recall:

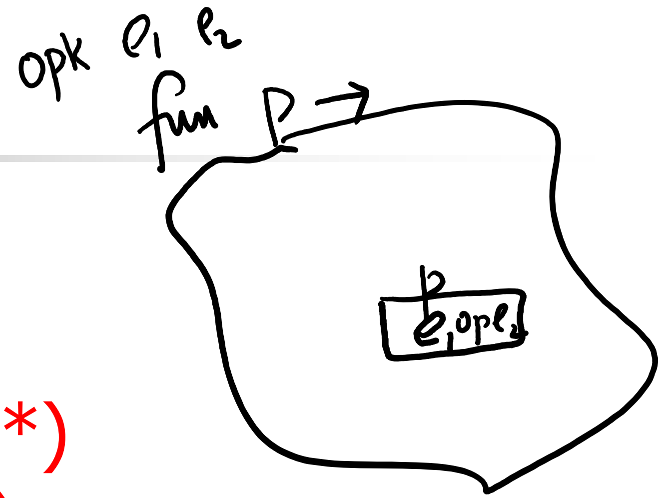
```
# let rec factorial n =
    if n = 0 then 1 else n * factorial (n - 1);;
  val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

Recursive Functions

```
# let rec factorial n =
    let b = (n = 0) in (* First computation *)
    if b then 1 (* Returned value *)
    else let s = n – 1 in (* Second computation *)
         let r = factorial s in  (* Third computation *)
         n * r in (* Returned value *) ;;
val factorial : int -> int = <fun>
# factorial 5;;
- : int = 120
```

let rec factorialk n k
eqk (n,0) fun b -> if b then k 1
else

Recursive Functions




```
# let rec factorialk n k =
   eqk (n, 0)
   (fun b ->  (* First computation *)
    if b then k 1 (* Passed value *)
    else subk (n, 1)  (* Second computation *)
    (fun s -> factorialk s  (* Third computation *)
    (fun r -> timesk (n, r) k))) (* Passed value *)
val factorialk : int -> int = <fun>
# factorialk 5 report;;
120
- : unit = ()
```

Recursive Functions

- To make recursive call, must build intermediate continuation to
  - take recursive value: r
  - build it to final result: n * r
  - And pass it to final continuation:
  - times (n, r) k = k (n * r)

Example: CPS for length

let rec length list = match list with [] -> 0

| (a :: bs) -> 1 + length bs

What is the let-expanded version of this?

```
let r = length bs in
let s = 1 + r
in s
```

```
lengthk list k
  = match list [] -> k 0
  | (a :: bs) -> lengthk bs (fun r -> addk (1, r) k)
```

Example: CPS for length

```
let rec length list = match list with [] -> 0
    | (a :: bs) -> 1 + length bs
```

What is the let-expanded version of this?

```
let rec length list = match list with [] -> 0
    | (a :: bs) -> let r1 = length bs in 1 + r1
```

Example: CPS for length

```
#let rec length list = match list with [] -> 0
    | (a :: bs) -> let r1 = length bs in 1 + r1
```

What is the CSP version of this?

Example: CPS for length

```
#let rec length list = match list with [] -> 0
    | (a :: bs) -> let r1 = length bs in 1 + r1;
```

What is the CSP version of this?

```
#let rec lengthk list k = match list with [ ] -> k 0
   | x :: xs -> lengthk xs (fun r -> addk (r,1) k);;
val lengthk : 'a list -> (int -> 'b) -> 'b = <fun>
# lengthk [2;4;6;8] report;;
4
- : unit = ()
```

Your turn now

Try Problem 12 on MP2

CPS for Higher Order Functions

- In CPS, every procedure / function takes a continuation to receive its result
- Procedures passed as arguments take continuations
- Procedures returned as results take continuations
- CPS version of higher-order functions must expect input procedures to take continuations

Example: all

let g5 n = n > 5

```
#let rec all (p, l) = match l with [] -> true
     | (x :: xs) -> let b = p x in
       if b then all (p, xs) else false
val all : ('a -> bool) * 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
let allk (pk, l) k = match l with [] -> k true
   | (x :: xs) -> pk x (fun b ->
                 if b then allk (pk, xs) k
                 else k false
```

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k =
```

Example: all

```
#let rec all (p, l) = match l with [] -> true
     | (x :: xs) -> let b = p x in
          if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->      true
```

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
```

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) ->
```

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
```

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
          (fun b -> if b then                    else
     )
```

Example: all

```
#let rec all (p, l) = match l with [] -> true
    | (x :: xs) -> let b = p x in
        if b then all (p, xs) else false
val all : ('a -> bool) -> 'a list -> bool = <fun>
```

- What is the CPS version of this?

```
#let rec allk (pk, l) k = match l with [] ->  k true
 | (x :: xs) -> pk x
          (fun b -> if b then allk (pk, xs) k else k
false)
val allk : ('a -> (bool -> 'b) -> 'b) * 'a list ->
(bool -> 'b) -> 'b = <fun>
```

Terms

- A function is in Direct Style when it returns its result back to the caller.
- A Tail Call occurs when a function returns the result of another function call without any more computations (eg tail recursion)
- A function is in Continuation Passing Style when it, and every function call in it, passes its result to another function.
- Instead of returning the result to the caller, we pass it forward to another function.

Terminology

- Tail Position: A subexpression s of expressions e, such that if evaluated, will be taken as the value of e
  - if (x>3) then x + 2 else x - 4
  - let x = 5 in x + 4
- Tail Call: A function call that occurs in tail position
  - if (h x) then f x else (x + g x)

Terminology

- Available: A function call that can be executed by the current expression
- The fastest way to be unavailable is to be guarded by an abstraction (anonymous function, lambda lifted).
  - if (h x) then f x else (x + g x)
  - if (h x) then (fun x -> f x) else (g (x + x))

Not available

CPS Transformation

- Step 1: Add continuation argument to any function definition:
  - let f arg = e $\Rightarrow$ let f arg k = e
  - Idea: Every function takes an extra parameter saying where the result goes
- Step 2: A simple expression in tail position should be passed to a continuation instead of returned:
  - return a $\Rightarrow$ k a
  - Assuming a is a constant or variable.
  - “Simple” = “No available function calls.”

CPS Transformation

- Step 3: Pass the current continuation to every function call in tail position
  - return f arg $\Rightarrow$ f arg k
  - The function "isn't going to return," so we need to tell it where to put the result.

CPS Transformation

- Step 4: Each function call not in tail position needs to be converted to take a new continuation (containing the old continuation as appropriate)
  - return op (f arg) $\Rightarrow$ f arg (fun r -> k(op r))
  - op represents a primitive operation
  - return f(g arg) $\Rightarrow$ g arg (fun r-> f r k)

Example

Before:

```
let rec add_list lst =
match lst with
  [ ] -> 0
| 0 :: xs -> add_list xs
| x :: xs -> (+) x
   (add_list xs);;
```

After:

```
let rec add_listk lst k =
                        (* rule 1 *)
match lst with
| [ ] -> k 0 (* rule 2 *)
| 0 :: xs -> add_listk xs k
                           (* rule 3 *)
| x :: xs -> add_listk xs
            (fun r -> k ((+) x r));;
                     (* rule 4 *)
```

CPS for sum

```
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
```

CPS for sum

```
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> let r1 = sum xs  in x + r1;;
```

CPS for sum

```
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
   | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
    | x :: xs -> sumk xs  (fun r1 -> addk x r1 k);;
```

CPS for sum

```
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> x + sum xs ;;
val sum : int list -> int = <fun>
# let rec sum list = match list with [ ] -> 0
    | x :: xs -> let r1 = sum xs  in x + r1;;
val sum : int list -> int = <fun>
# let rec sumk list k = match list with [ ] -> k 0
     | x :: xs -> sumk xs  (fun r1 -> addk (x, r1) k);;
val sumk : int list -> (int -> 'a) -> 'a = <fun>
# sumk [2;4;6;8] report;;
20
- : unit = ()
```

Other Uses for Continuations

- CPS designed to preserve order of evaluation
- Continuations used to express order of evaluation
- Can be used to change order of evaluation
- Implements:
  - Exceptions and exception handling
  - Co-routines
  - (pseudo, aka green) threads

Exceptions - Example

```
# exception Zero;;
exception Zero
# let rec list_mult_aux list =
    match list with [ ] -> 1
    | x :: xs ->
     if x = 0 then raise Zero
                 else x * list_mult_aux xs;;
val list_mult_aux : int list -> int = <fun>
```

Exceptions - Example

```
# let list_mult list =
    try list_mult_aux list with Zero -> 0;;
val list_mult : int list -> int = <fun>
# list_mult [3;4;2];;
- : int = 24
# list_mult [7;4;0];;
- : int = 0
# list_mult_aux [7;4;0];;
Exception: Zero.
```

Exceptions

- When an exception is raised
  - The current computation is aborted
  - Control is “thrown” back up the call stack until a matching handler is found
  - All the intermediate calls waiting for a return values are thrown away

Implementing Exceptions

```
# let multkp (m, n) k =
   let r = m * n in
    (print_string "product result: ";
     print_int r; print_string "\n";
     k r);;
val multkp : int ( int -> (int -> 'a) -> 'a =
  <fun>
```

Implementing Exceptions

```
# let rec list_multk_aux list k kexcp =
   match list with [ ] -> k 1
   | x :: xs -> if x = 0 then  kexcp  0
     else list_multk_aux xs
              (fun r -> multkp (x, r) k) kexcp;;
val list_multk_aux : int list -> (int -> 'a) -> (int -> 'a)
  -> 'a = <fun>
# let rec list_multk list k = list_multk_aux list  k  k;;
val list_multk : int list -> (int -> 'a) -> 'a = <fun>
```

Implementing Exceptions

```
# list_multk [3;4;2] report;;
product result: 2
product result: 8
product result: 24
24
- : unit = ()
# list_multk [7;4;0] report;;
0
- : unit = ()
```