## Probability Refresher and Cycle Analysis

## A Quick Probability Refresher

- A random variable, $X$, can take on a number of different possible values
- Example: the number of pigeons on the windowsill outside is a random variable with possible values $1,2,3, \ldots$
- Each time we observe (or sample) the random variable, it may take on a different value


## A Quick Probability Refresher

- A random variable takes on each of these values with a specified probability
- Example: $X=\{0,1,2,3,4\}$
- $P[X=0]=.1, P[X=1]=.2, P[X=2]=.4$, $P[X=3]=.1, P[X=4]=.2$
- The sum of the probabilities of all values equals 1
- $\Sigma_{\text {all values }} P[X=$ value $]=1$


## A Quick Probability Refresher

- Example
- Suppose we throw two dice and the random variable, $X$, is the sum of the two dice
- Possible values of $X$ are $\{2,3,4,5,6,7,8,9,10,11,12\}$
- $P[X=2]=P[X=12]=1 / 36$
$P[X=3]=P[X=11]=2 / 36$
$P[X=4]=P[X=10]=3 / 36$
$P[X=5]=P[X=9]=4 / 36$
- $P[X=6]=P[X=8]=5 / 36$
- $P[X=7]=6 / 36$

$$
\text { Note: } \sum_{i=2}^{12} P[X=i]=1
$$

## A Quick Probability Refresher

- Expected Value
- Can be thought of a "long term average" of observing the random variable a large number of times

$$
E[X]=\bar{x}=\sum_{\substack{\text { Al possible } \\ \text { values of } x}} \text { Value * } P[X=\text { value }
$$

- Example: dice - $E[X]$

$$
\begin{aligned}
= & 2 * 1 / 36+3 * 2 / 36+4^{*} 3 / 36+5^{*} 4 / 36+6^{*} 5 / 36+ \\
& 7 * 6 / 36+8 * 5 / 36+94 / 36+10^{*} 3 / 36+11^{*} 2 / 36+ \\
& 12^{*} 1 / 36
\end{aligned}
$$

## Probability Example

- Basic probability notions
- Two useful rules
- Probabilities of all possible events sum to 1
- Probability of independent events
- Product of probabilities of events
- e.g., probability of two coins coming up heads

$$
=1 / 2 \times 1 / 2=1 / 4
$$

- Calculating averages/expected values
- Function $f$
- Multiply $f$ by probability for each possible event
- Sum over all events


## Probability Example - Problem

- Given a bag with $N$ balls
- 1 blue ball
- N-1 white balls
- Algorithm
- pick a ball
- if blue, you win
- else return to bag
- repeat $N$ times
- Question
- What is your chance of winning for large N?


## Probability Example - Solution

- Can write as a sum
- Chance of finding blue on first try $=1 / \mathrm{N}$
- On second try $=[(N-1) / N]$ * $(1 / N)$
- Etc.
- Instead, write
- 1-(chance of losing)
- Parenthesized term
- Product of $N$ factors
- Each factor $=(N-1) / N$
- $1-[(N-1) / N]^{N}$


## Probability Example - Solution

- For $N=2$,
- $1 / 2$ first is white
- $1 / 2$ second is white
- $1 / 4$ both are white
- $3 / 4$ chance to win $=1-(1 / 2)^{2}$
- For $N=3$,
- $2 / 3$ first is white
- $2 / 3$ second is white
- $2 / 3$ third is white
- 8/27 all three are white
- $19 / 27$ chance to win $=1-(2 / 3)^{3} \quad(<3 / 4)$


## Probability Example - Solution

- N=4 probability of win $=68 \%$
- $N=5$ probability of win $=67 \%$
- N=8 probability of win $=66 \%$
- large $N$ ? 0 ?

$$
\lim _{N \rightarrow \infty}\left(\frac{N-1}{N}\right)^{N}
$$

## Fun Example

- Flip a coin repeatedly.
- Two heads in a row scores 1 point.
- Scoring pairs may not overlap
- (e.g., three heads in a row does not score 2 points).
- On average, how many points do you score per flip?


## A Different Example

- What fraction of time (on average) is spent in state E?



## Cycle Analysis

- Start with a discrete Markov process
- Transitions happen periodically (every $\Delta \mathrm{t}$ )
- Probabilities independent of past/future behavior
- Form all possible cyclic sequences (cycles)
- Pick a "start" state
- List all cycles from that state
- Calculate probability per cycle
- Calculate average cycle length
- Can calculate expected values of cycle-dependent properties with average length and cycle probabilities


## Example

cycle
probability


## Example

cycle ABS

CBS

CDES

average cycle length
$\overline{\mathrm{ABS}} \overline{\mathrm{CBS}} \quad \overline{\mathrm{CDES}}$

## Example



- dividing by average length...

$$
=0.125 / 3.125=0.04
$$

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## Fun Example

- cycle probability
- T
- HT

1/2
1/4

- HH

1/4
average cycle length
average score per cycle average score per flip

## Fun Example

- cycle probability
- T
- HT
- HH

1/2
1/4
1/4
average cycle length
average score per cycle average score per flip

$=1 / 2+1 / 2+1 / 2=3 / 2$ flips
$=1 / 4$ points
$=(1 / 4) /(3 / 2)=1 / 6 \mathrm{pts} / \mathrm{flip}$

