CS 473: Algorithms

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Introduction to Randomized Algorithms: QuickSort

Lecture 7
Feb 6, 2018

Most slides are courtesy Prof. Chekuri

Outline

Randomization is very powerful

How do you play R-P-S?

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Calculating insurance.

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Randomization is very powerful

How do you play R-P-S?
Calculating insurance.

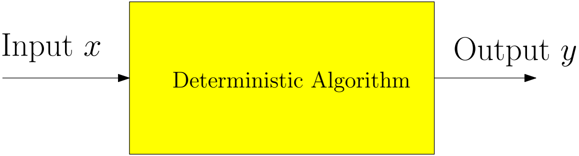
Our goal

- Basics of randomization – probability space, expectation, events, random variables, etc.
- Randomized Algorithms – Two types
  - Las Vegas
  - Monte Carlo
- Randomized Quick Sort

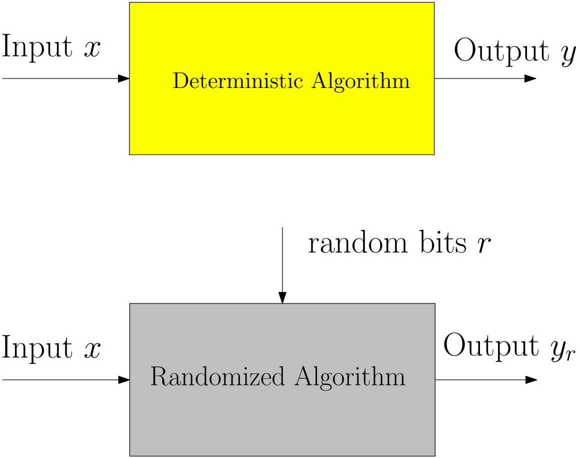
Part I

Introduction to Randomized Algorithms

Randomized Algorithms



Randomized Algorithms



Example: Verifying Matrix Multiplication

Problem

Given three $n \times n$ matrices A, B, C is $AB = C$?

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1. Multiply A and B and check if equal to C.
2. Running time?

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Given three $n \times n$ matrices A, B, C is $AB = C$?

Deterministic algorithm:

1. Multiply A and B and check if equal to C.
2. Running time? $O(n^3)$ by straight forward approach. $O(n^{2.37})$ with fast matrix multiplication (complicated and impractical).

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Randomized algorithm:

1. Pick a random $n \times 1$ vector r.
2. Return the answer of the equality $ABr = Cr$.
3. Running time?

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Randomized algorithm:

1. Pick a random $n \times 1$ vector r.
2. Return the answer of the equality $ABr = Cr$.
3. Running time? $O(n^2)$!

Theorem

If $AB = C$ then the algorithm will always say YES. If $AB \neq C$ then the algorithm will say YES with probability at most $1/2$. Can repeat the algorithm 100 times independently to reduce the probability of a false positive to $1/2^{100}$.

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3. Often randomized algorithms are (much) simpler and/or more efficient.
4. Several deep connections to mathematics, physics etc.
5. ...
6. Lots of fun!

Average case analysis vs Randomized algorithms

Average case analysis:

1. Fix a deterministic algorithm.
2. Assume inputs comes from a probability distribution.
3. Analyze the algorithm's *average* performance over the distribution over inputs.

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Randomized algorithms:

1. Input is arbitrary (worst case).
2. Algorithm uses random bits, and therefore on each input the behavior of the algorithm is random.
3. Analyze algorithms *average* performance over any given (worst case) input where the average is over the random bits that the algorithm uses.

Part II

Basics of Discrete Probability

Discrete Probability

We restrict attention to finite probability spaces.

Definition

A discrete probability space is a pair $(\Omega, \Pr)$ consists of finite set Ω of **elementary events** and function $\Pr[:] \Omega \rightarrow [0, 1]$ which assigns a probability $\Pr[\omega]$ for each $\omega \in \Omega$ such that $\sum_{\omega \in \Omega} \Pr[\omega] = 1$.

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Example

An unbiased coin. $\Omega = \{H, T\}$ and $\Pr[H] = \Pr[T] = 1/2$.

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for $1 \leq i \leq 6$.

Events

Definition

Given a probability space $(\Omega, \Pr)$ an **event** is a subset of Ω. In other words an event is a collection of elementary events. The probability of an event $A \subseteq \Omega$, denoted by $\Pr[A]$, is $\sum_{\omega \in A} \Pr[\omega]$.

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$\Pr[A] = |A|/36 = 1/2$.

Independent Events

Definition

Given a probability space $(\Omega, \Pr)$ and two events A, B are **independent** if and only if $\Pr[A \cap B] = \Pr[A] \Pr[B]$. Otherwise they are *dependent*. In other words A, B independent implies one does not affect the other.

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2. A : both are not tails. B : second coin is heads. $\Pr[A] =$

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2. A : both are not tails. B : second coin is heads. $\Pr[A] = 3/4$, $\Pr[B] = 1/2$, $\Pr[A \cap B] = 1/2$. **dependent**.

Union bound

The probability of the union of two events, is no bigger than the sum of their probabilities.

Lemma

For any two events $\mathcal{E}$ and $\mathcal{F}$, we have that
$\mathbf{Pr}\left[\mathcal{E} \cup \mathcal{F}\right] \leq \mathbf{Pr}\left[\mathcal{E}\right] + \mathbf{Pr}\left[\mathcal{F}\right]$.

Proof.

Consider $\mathcal{E}$ and $\mathcal{F}$ to be a collection of elmentery events (which they are). We have

$$\mathbf{Pr}\left[\mathcal{E} \cup \mathcal{F}\right] = \sum_{x \in \mathcal{E} \cup \mathcal{F}} \mathbf{Pr}[x]$$
$$\leq \sum_{x \in \mathcal{E}} \mathbf{Pr}[x] + \sum_{x \in \mathcal{F}} \mathbf{Pr}[x] = \mathbf{Pr}\left[\mathcal{E}\right] + \mathbf{Pr}\left[\mathcal{F}\right].$$

Random Variables

Definition (Random Variable)

Given a probability space $(\Omega, \Pr)$ a (real-valued) random variable X over Ω is a function that maps each elementary event to a real number. In other words $X : \Omega \to \mathbb{R}$.

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Definition (Expectation)

For a random variable X over a probability space $(\Omega, \Pr)$ the **expectation** of X is defined as $\sum_{\omega \in \Omega} \Pr[\omega] \, X(\omega)$. In other words, the expectation is the average value of X according to the probabilities given by $\Pr[\cdot]$.

Expectation

Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for each $i \in \Omega$.

1. $X : \Omega \to \mathbb{R}$ where $X(i) = i \mod 2 \in \{0, 1\}$.

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2. $Y : \Omega \to \mathbb{R}$ where $Y(i) = i^2$.

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2. $Y : \Omega \to \mathbb{R}$ where $Y(i) = i^2$. Then $\mathbf{E}[Y] = \sum_{i=1}^{6} \frac{1}{6} \cdot i^2 = 91/6$.

Expected number of vertices?

Let $G = (V, E)$ be a graph with n vertices and m edges. Let H be the graph resulting from independently deleting every vertex of G with probability $1/2$. Compute the expected number of vertices in H.

(A) n/2.

(B) n/4.

(C) m/2.

(D) m/4.

(E) none of the above.

Expected number of vertices is:

Probability Space

- $\Omega = \{0,1\}^n$. For $\omega \in \{0,1\}^n$, $\omega_v = 1$ if vertex v is present in H, else is zero.
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$$
\begin{aligned}
\mathbf{E}[X] &= \sum_{\omega\in\Omega} \Pr[\omega]\, X(\omega) \\
&= \sum_{\omega\in\Omega} 1/2^n X(\omega) \\
&= 1/2^n \sum_{k=0}^{n} \binom{n}{k} k \\
&= 1/2^n (2^n \tfrac{n}{2}) \\
&= n/2
\end{aligned}
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Expected number of edges?

Let $G = (V, E)$ be a graph with n vertices and m edges. Let H be the graph resulting from independently deleting every vertex of G with probability $1/2$. The expected number of edges in H is

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How to compute $\mathbf{E}[X]$?

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Special type of random variables that are quite useful.

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Given a probability space $(\Omega, \Pr)$ and an event $A \subseteq \Omega$ the **indicator random variable** X_A is a binary random variable where $X_A(\omega) = 1$ if $\omega \in A$ and $X_A(\omega) = 0$ if $\omega \notin A$.

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Example

A 6-sided unbiased die. $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr[i] = 1/6$ for each $i \in \Omega$. Let A be the even that i is divisible by 3, i.e., $A = \{3, 6\}$. Then $X_A(i) = 1$ if $i \in \{3, 6\}$ and 0 otherwise.

Expectation

Proposition

For an indicator variable X_A, $\mathbf{E}[X_A] = \mathbf{Pr}[A]$.

Proof.

$$\begin{aligned}
\mathbf{E}[X_A] &= \sum_{\omega \in \Omega} X_A(\omega) \, \mathbf{Pr}[\omega] \\
&= \sum_{\omega \in A} 1 \cdot \mathbf{Pr}[\omega] + \sum_{\omega \in \Omega \setminus A} 0 \cdot \mathbf{Pr}[\omega] \\
&= \sum_{\omega \in A} \mathbf{Pr}[\omega] \\
&= \mathbf{Pr}[A] \,.
\end{aligned}$$

$\square$

Linearity of Expectation

Lemma

Let X, Y be two random variables (not necessarily independent) over a probability space $(\Omega, \Pr)$. Then $\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$.

Proof.

$$\begin{aligned}\mathbf{E}[X+Y] &= \sum_{\omega \in \Omega} \Pr[\omega]\,(X(\omega) + Y(\omega)) \\ &= \sum_{\omega \in \Omega} \Pr[\omega]\,X(\omega) + \sum_{\omega \in \Omega} \Pr[\omega]\,Y(\omega) = \mathbf{E}[X] + \mathbf{E}[Y].\end{aligned}$$

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$$= \sum_{\omega \in \Omega} \Pr[\omega] X(\omega) + \sum_{\omega \in \Omega} \Pr[\omega] Y(\omega) = \mathbf{E}[X] + \mathbf{E}[Y].$$

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Corollary

$\mathbf{E}[a_1 X_1 + a_2 X_2 + \ldots + a_n X_n] = \sum_{i=1}^{n} a_i \mathbf{E}[X_i]$.

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It is important to setup random variables carefully.

Expected number of triangles?

Let $G = (V, E)$ be a graph with n vertices and m edges. Assume G has t triangles (i.e., a triangle is a simple cycle with three vertices). Let H be the graph resulting from deleting independently each vertex of G with probability $1/2$. The expected number of triangles in H is

- **(A)** t/2.
- **(B)** t/4.
- **(C)** t/8.
- **(D)** t/16.
- **(E)** none of the above.

Independent Random Variables

Definition

Random variables X, Y are said to be independent if

$$\forall x, y \in \mathbb{R}, \quad \mathbf{Pr}[X = x \wedge Y = y] = \mathbf{Pr}[X = x]\,\mathbf{Pr}[Y = y]$$

.

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Examples

Two independent un-biased coin flips: $\Omega = \{HH, HT, TH, TT\}$.

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- $X = 1$ if first coin is H else 0. $Y = 1$ if second coin is H else 0. Independent.
- $X = \#H$, $Y = \#T$. Dependent. Why?

Independent Random Variables

Lemma

If X and Y are independent then $\mathbf{E}[X \cdot Y] = \mathbf{E}[X] \cdot \mathbf{E}[Y]$

Proof.

$$\begin{aligned}
\mathbf{E}[X \cdot Y] &= \sum_{\omega \in \Omega} \Pr[\omega] \left(X(\omega) \cdot Y(\omega)\right) \\
&= \sum_{x,y \in \mathbb{R}} \Pr[X = x \wedge Y = y] \left(x \cdot y\right) \\
&= \sum_{x,y \in \mathbb{R}} \Pr[X = x] \cdot \Pr[Y = y] \cdot x \cdot y \\
&= \left(\sum_{x \in \mathbb{R}} \Pr[X = x]\, x\right)\left(\sum_{y \in \mathbb{R}} \Pr[Y = y]\, y\right) = \mathbf{E}[X]\, \mathbf{E}[Y]
\end{aligned}$$

□

Types of Randomized Algorithms

Typically one encounters the following types:

1. **Las Vegas randomized algorithms:** for a given input x output of *algorithm is always correct* but the *running time is a random variable*. In this case we are interested in analyzing the *expected* running time.

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2. **Monte Carlo randomized algorithms:** for a given input x the *running time is deterministic* but the *output is random; correct with some probability*. In this case we are interested in analyzing the *probability* of the correct output (and also the running time).
3. Algorithms whose running time and output may both be random.

Analyzing Las Vegas Algorithms

Deterministic algorithm Q for a problem Π:

1. Let $Q(x)$ be the time for Q to run on input x of length $|x|$.
2. Worst-case analysis: run time on worst input for a given size n.

$$T_{wc}(n) = \max_{x:|x|=n} Q(x).$$

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$$T_{rand-wc}(n) = \max_{x:|x|=n} \mathbf{E}[R(x)].$$

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Randomized algorithm M for a problem Π:

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4. Worst-case analysis: success probability on worst input

$$P_{rand-wc}(n) = \min_{x:|x|=n} \Pr[x].$$

Part III

Why does randomization help?

Ping and find.

Consider a deterministic algorithm A that is trying to find an element in an array X of size n. At every step it is allowed to ask the value of one cell in the array, and the adversary is allowed after each such ping, to shuffle elements around in the array in any way it seems fit. For the best possible deterministic algorithm the number of rounds it has to play this game till it finds the required element is

(A) $O(1)$

(B) $O(n)$

(C) $O(n \log n)$

(D) $O(n^2)$

(E) ∞.

Ping and find randomized.

Consider an algorithm **randFind** that is trying to find an element in an array X of size n. At every step it asks the value of one random cell in the array, and the adversary is allowed after each such ping, to shuffle elements around in the array in any way it seems fit. This algorithm would stop in expectation after

- **(A)** $O(1)$
- **(B)** $O(\log n)$
- **(C)** $O(n)$
- **(D)** $O(n^2)$
- **(E)** ∞.

steps.

Abundance of witnesses

Consider the problem of finding an "approximate median" of an unsorted array $A[1..n]$: an element of A with rank between $n/4$ and $3n/4$.

- Finding an approximate median is not any easier than a proper median.

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- Finding an approximate median is not any easier than a proper median.
- $n/2$ elements of A qualify as approximate medians and hence a random element is good with probability $1/2$!

Part IV

Randomized Quick Sort

QuickSort

Deterministic **QuickSort**

1. Pick a pivot element from array
2. Split array into 3 subarrays: those smaller than pivot, those larger than pivot, and the pivot itself.
3. Recursively sort the subarrays, and concatenate them.

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1. Pick a pivot element **uniformly at random** from the array
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Note: On *every* input randomized **QuickSort** takes $O(n \log n)$ time in expectation. On *every* input it may take $\Omega(n^2)$ time with some small probability.

Randomized **QuickSort**

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1. Pick a pivot element *uniformly at random* from the array.
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What random variables to define?
What are the events of the algorithm?

Analysis via Recurrence

1. Given array A of n *distinct* numbers.
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Since each element of A has probability exactly of $1/n$ of being chosen:

$$\mathbf{E}[X_i] = \mathbf{Pr}[\text{pivot has rank } i] = 1/n.$$

Independence of Random Variables

Lemma

Random variables X_i *is independent of random variables* $Q(A^i_{left})$ *as well as* $Q(A^i_{right})$, i.e.

$$\mathbf{E}\left[X_i \cdot Q(A^i_{left})\right] = \mathbf{E}[X_i]\,\mathbf{E}\left[Q(A^i_{left})\right]$$
$$\mathbf{E}\left[X_i \cdot Q(A^i_{right})\right] = \mathbf{E}[X_i]\,\mathbf{E}\left[Q(A^i_{right})\right]$$

Proof.

This is because the algorithm, while recursing on $Q(A^i_{\text{left}})$ and $Q(A^i_{\text{right}})$ uses new random coin tosses that are independent of the coin tosses used to decide the first pivot. Only the latter decides value of X_i. □

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Let $T(n) = \max_{A:|A|=n} \mathbf{E}[Q(A)]$ be the worst-case expected running time of randomized **QuickSort** on arrays of size n.

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Note that above holds for any A of size n. Therefore

$$\max_{A:|A|=n} \mathbf{E}[Q(A)] = T(n) \leq n + \sum_{i=1}^{n} \frac{1}{n} \left(T(i-1) + T(n-i) \right).$$

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$$T(n) \le n + \sum_{i=1}^{n} \frac{1}{n}\left(T(i-1) + T(n-i)\right)$$

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Proof.

(Guess and) Verify by induction. □

Part V

Slick analysis of QuickSort

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$$Q(A) = \sum_{1 \leq i < j \leq n} X_{ij}$$

and hence by linearity of expectation,

$$\mathbf{E}\Big[Q(A)\Big] = \sum_{1 \leq i < j \leq n} \mathbf{E}\Big[X_{ij}\Big] = \sum_{1 \leq i < j \leq n} \mathbf{Pr}\Big[R_{ij}\Big].$$

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With ranks:	6	4	8	1	2	3	7	5

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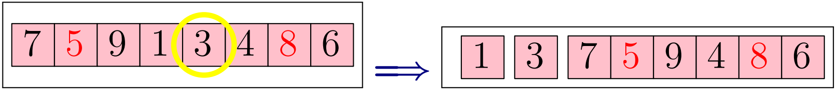

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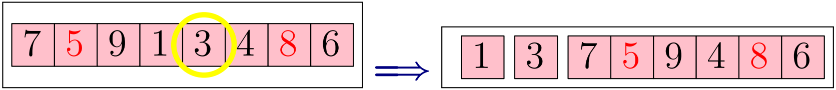

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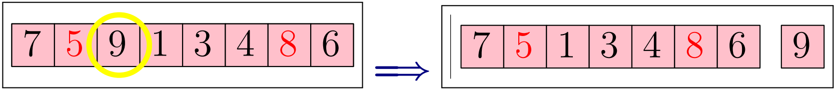

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$\Longrightarrow$

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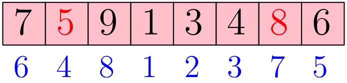

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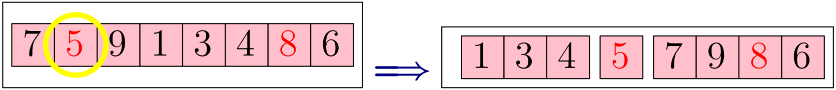
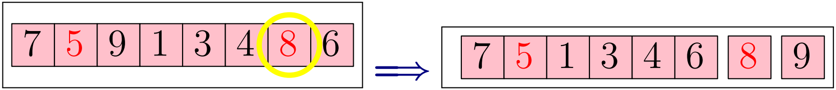


2. If pivot is **8** (rank 7). Bingo!



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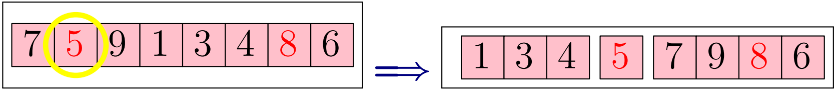

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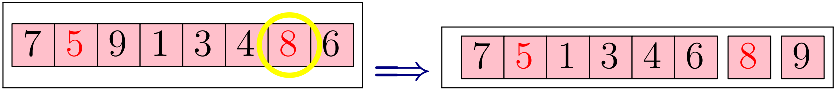
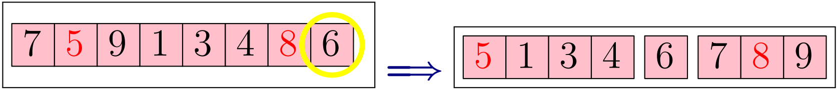


3. If pivot in between the two numbers (say **6** [rank 5]):



5 and **8** will never be compared to each other.

A Slick Analysis of **QuickSort**

Question: What is $\mathbf{Pr[R_{i,j}]}$?

Conclusion:

$R_{i,j}$ happens if and only if:

ith or jth ranked element is the first pivot out of ith to jth ranked elements.

$$\mathbf{Pr}[R_{i,j}] = Pr[i\text{th or } j\text{th ranked element is the pivot} \mid \text{pivot has rank in } \{i, i+1 \ldots, j-1, j\}]$$

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There are $k = j - i + 1$ relevant elements.

$$\mathbf{Pr}\left[R_{i,j}\right] = \frac{2}{k} = \frac{2}{j - i + 1}.$$

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Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$

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Observation: If pivot is chosen outside S then all of S either in left array or right array.
Observation: a_i and a_j separated when a pivot is chosen from S for the first time. Once separated no comparison.
Observation: a_i is compared with a_j if and only if either a_i or a_j is chosen as a pivot from S at separation... □

A Slick Analysis of **QuickSort**

Continued...

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Proof.

Let $a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n$ be sort of A. Let $S = \{a_i, a_{i+1}, \ldots, a_j\}$

Observation: a_i is compared with a_j if and only if either a_i or a_j is chosen as a pivot from S at separation.

Observation: Given that pivot is chosen from S the probability that it is a_i or a_j is exactly $2/|S| = 2/(j - i + 1)$ since the pivot is chosen uniformly at random from the array. □

A Slick Analysis of **QuickSort**

Continued...

$$\mathbf{E}\Big[Q(A)\Big] = \sum_{1 \le i < j \le n} \mathbf{E}[X_{ij}] = \sum_{1 \le i < j \le n} \Pr[R_{ij}] \,.$$

Lemma

$\Pr[R_{ij}] = \frac{2}{j-i+1}$.

A Slick Analysis of **QuickSort**

Continued...

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A Slick Analysis of **QuickSort**

Continued...

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A Slick Analysis of **QuickSort**

Continued...

Lemma

$\Pr[R_{ij}] = \frac{2}{j-i+1}$.

$$
\begin{aligned}
\mathbf{E}\Big[Q(A)\Big] &= \sum_{1\le i<j\le n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}
\end{aligned}
$$

A Slick Analysis of **QuickSort**

Continued...

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$\Pr[R_{ij}] = \frac{2}{j-i+1}$.

$$\mathbf{E}\left[Q(A)\right] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

A Slick Analysis of **QuickSort**

Continued...

Lemma

$\mathbf{Pr}[R_{ij}] = \frac{2}{j-i+1}$.

$$\mathbf{E}\Big[Q(A)\Big] = 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{j-i+1}$$

A Slick Analysis of **QuickSort**

Continued...

Lemma

$\Pr[R_{ij}] = \frac{2}{j-i+1}$.

$$\mathbf{E}\Big[Q(A)\Big] = 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{1}{j-i+1} = 2\sum_{i=1}^{n-1}\quad\sum_{\Delta=2}^{n-i+1}\frac{1}{\Delta}$$

A Slick Analysis of **QuickSort**

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$$\leq 2\sum_{i=1}^{n-1}(H_{n-i+1}-1) \;\leq\; 2\sum_{1\leq i<n} H_n$$

$$H_k = \sum_{i=1}^{k}\frac{1}{i} = \Theta(\log k)$$

A Slick Analysis of **QuickSort**

Continued...

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$$\leq 2\sum_{i=1}^{n-1}(H_{n-i+1}-1) \;\leq\; 2\sum_{1\leq i<n} H_n$$

$$\leq 2nH_n = O(n\log n)$$

$H_k = \sum_{i=1}^{k}\frac{1}{i} = \Theta(\log k)$

Where do I get random bits?

Question: Are true random bits available in practice?

1. Buy them!
2. CPUs use physical phenomena to generate random bits.
3. Can use pseudo-random bits or semi-random bits from nature. Several fundamental unresolved questions in complexity theory on this topic. Beyond the scope of this course.
4. In practice pseudo-random generators work quite well in many applications.
5. The model is interesting to think in the abstract and is very useful even as a theoretical construct. One can *derandomize* randomized algorithms to obtain deterministic algorithms.