CS 473: Algorithms

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Spring 2018

High Probability Analysis & Universal Hashing

Lecture 09

Feb 13, 2018

Most slides are courtesy Prof. Chekuri

Outline

Randomized **QuickSort** w.h.p. (any questions?)

What is the probability that the algorithm will terminate in $O(n \log n)$ time?

Balls & Bins

- Expected bin size.
- Expected max bin size $\rightarrow$ max size w.h.p.
- Analogy to hashing

Hashing

Part I

Randomized **QuickSort** (Contd.)

Randomized **QuickSort**: Recall

Input: Array A of n distinct numbers. **Output:** Numbers in sorted order.

Randomized **QuickSort**

1. Pick a pivot element *uniformly at random* from A.
2. Split array into 2 subarrays: those smaller than pivot (L), and those larger than pivot (R).
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Question: With what probability it takes $O(n \log n)$ time?

Randomized **QuickSort**: High Probability Analysis

Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32 n \ln n] \geq 1 - 1/n^3$.

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If $n = 100$ then this gives $\Pr[Q(A) \leq 32n \ln n] \geq 0.99999$.

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Outline of the proof

- If depth of recursion is k then $Q(A) \leq kn$.
- Prove that depth of recursion $\leq 32 \ln n$ with high probability (w.h.p.) . This will imply the result.

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  1. Focus on a single element. Prove that it "participates" in $> 32 \ln n$ levels with probability (w.p.) at most $1/n^4$.
  2. By union bound, any of the n elements participates in $> 32 \ln n$ levels w.p. at most

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  1. Focus on a single element. Prove that it "participates" in $> 32 \ln n$ levels with probability (w.p.) at most $1/n^4$.
  2. By union bound, any of the n elements participates in $> 32 \ln n$ levels w.p. at most $1/n^3$.
  3. Therefore, all elements participate in $\leq 32 \ln n$ w.p. $(1 - 1/n^3)$.

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An element participates in $> 32 \ln n$ w.p. $\leq 1/n^4$.

Intuition

1. When we pick a pivot from an array of size n uniformly at random, what is the probability that its rank is between $n/4$ and $3n/4$?

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3. If an array is reduced to at least its $3/4$th size every time, then after how many rounds only one element remains? $\leq 4 \ln n$.
4. If $32 \ln n$ splits, then $\mathbf{E}[\text{Balanced-split}] = 16 \ln n$. Out of these there are $< 4 \ln n$ balanced split w.p. $\leq {}^1/_{n^4}$.

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- For $|S_k| = 1$, $\rho = 4 \ln n \geq \log_{4/3} n$ suffices.

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(\textit{Chernoff}) \quad &\leq 2e^{\frac{-\delta^2 \mu}{2}} \\
&= 2e^{-\frac{9k}{64}} \\
&= 2e^{-4.5 \ln n} \leq \frac{1}{n^4}
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Theorem

With high probability (i.e., $1 - \frac{1}{n^3}$*) the depth of the recursion of* **QuickSort** *is* $\leq 32 \ln n$*. Due to* n *comparisons in each level, with high probability, the running time of* **QuickSort** *is* $O(n \ln n)$.

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Q: How to increase the probability?

Part II

Balls and Bins

Expected Bin Size

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- Too many to count!!

Expected Max Bin Size (Contd.)

Problem

What is the expected maximum bin size?

R.V. $Z = \max_{j=1}^{n} Y_j$. Show $\mathbf{E}[Z] \leq O\left(\frac{\ln n}{\ln \ln n}\right)$?

Possible Solution

- If $\Pr\left[Z > \frac{8 \ln n}{\ln \ln n}\right] \leq 1/n^2$, then: define $A = \frac{8 \ln n}{\ln \ln n}$.

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&\le A \cdot \Pr[Z \le A] + n \cdot \Pr[Z > A] \\
&\le A \cdot (1) + n \cdot (1/n^2) = O(A) = O\left(\frac{\ln n}{\ln \ln n}\right)
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Bound $\Pr\left[Z > \frac{8\ln n}{\ln\ln n}\right]$.

Expected Max Bin Size (Contd.)

Bound $\mathbf{Pr}\left[Z > \frac{8 \ln n}{\ln \ln n}\right]$ using Chernoff inequality.

Chernoff Ineq. We Saw

$X_1, \ldots, X_k$ independent binary R.V., and $X = \sum_{i=1}^{k} X_i$,
$\mu = \mathbf{E}[X]$, then for $0 < \delta < 1$

$$\mathbf{Pr}[X \geq (1+\delta)\mu] \leq e^{-\delta^2\mu/3} \quad \& \quad \mathbf{Pr}[X \leq (1-\delta)\mu] \leq e^{-\delta^2\mu/2}$$

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Stronger Versions

- For $\delta > 0$, $\Pr[X > (1+\delta)\mu] < \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$.
- For $0 < \delta < 1$ $\Pr[X < (1-\delta)\mu] < \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$

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Solution

- Recall: $Y_j =\#$ balls in bin j, $\mathbf{E}[Y_j] = 1$, and $A = \frac{8 \ln n}{\ln \ln n}$

$$\Pr[Y_j > A] = \Pr[Y_j \ge A\,\mathbf{E}[Y]] < \left(\frac{e^{A-1}}{A^A}\right) < \left(\frac{n^{6/\ln \ln n}}{A^A}\right)$$

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$$\Pr\left[Y_j > \frac{8 \ln n}{\ln \ln n}\right] < 1/n^3$$

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- Recall: $Y_j =\#$ balls in bin j. $\mathbf{E}[Y_j] = 1$.

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Expected Max Bin Size (Contd.)

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What is the expected maximum bin size? Let $Z = \max_{j=1}^{n} Y_j$.

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$\Omega(\frac{\ln n}{\ln \ln n})$ is a lower bound as well!

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Storing elements in a table such that look up is $O(1)$-time.

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Part III

Hash Tables

Dictionary Data Structure

1. $\mathcal{U}$: universe of keys with total order: numbers, strings, etc.
2. Data structure to store a subset $S \subseteq \mathcal{U}$
3. **Operations:**
   1. **Search/lookup**: given $x \in \mathcal{U}$ is $x \in S$?
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5. **Dynamic** structure: S changes rapidly so inserts and deletes as important as lookups.

Dictionary Data Structures

Common solutions:

1. Static:
    1. Store S as a *sorted* array
    2. **Lookup**: Binary search in $O(\log |S|)$ time (comparisons)
2. Dynamic:
    1. Store S in a *balanced* binary search tree
    2. Lookup, Insert, Delete in $O(\log |S|)$ time (comparisons)

Dictionary Data Structures

Question: "Should Tables be Sorted?"
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Hashing is a widely used & powerful technique for dictionaries.

Motivation:

1. Universe $\mathcal{U}$ may not be (naturally) totally ordered.
2. Keys correspond to large objects (images, graphs etc) for which comparisons are very expensive.
3. Want to improve "average" performance of lookups to $O(1)$ even at cost of extra space or errors with small probability: many applications for fast lookups in networking, security, etc.

Hashing and Hash Tables

Hash Table data structure:

1. A (hash) table/array T of size m (the table **size**).
2. A hash function $h : \mathcal{U} \rightarrow \{0, \ldots, m-1\}$.
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Ideal situation:

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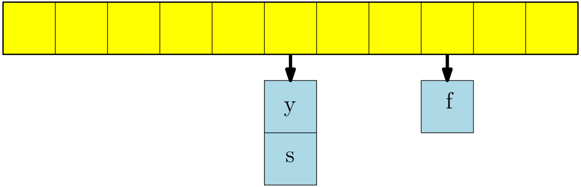
Collisions unavoidable if $|T| < |\mathcal{U}|$. Several techniques to handle them.

Handling Collisions: Chaining

Collision: $h(x) = h(y)$ for some $x \neq y$.

Chaining to handle collisions:

1. For each slot i store all items hashed to slot i in a linked list. $T[i]$ points to the linked list
2. **Lookup**: to find if $y \in \mathcal{U}$ is in T, check the linked list at $T[h(y)]$. Time proportion to size of linked list.



This is also known as **Open hashing**.

Handling Collisions

Several other techniques:

1. Cuckoo hashing.
   Every value has two possible locations. When inserting, insert in one of the locations, otherwise, kick stored value to its other location. Repeat till stable. if no stability then rebuild table.
2. ...
3. Others.

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Questions:

1. Complexity of evaluating h on a given element?
2. Relative sizes of the universe $\mathcal{U}$ and the set to be stored S.
3. Size of table relative to size of S.
4. Worst-case vs average-case vs randomized (expected) time?
5. How do we choose h?

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Main and interrelated questions:

1. Worst-case vs average-case vs randomized (expected) time?
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Lesson: For every hash function there is a very bad set. Bad set. Bad.

How many hash functions are there, anyway?

Let $\mathcal{H}$ be the set of all functions from $\mathcal{U} = \{1, \ldots, U\}$ to $\{1, \ldots, m\}$. The number of functions in $\mathcal{H}$ is

- **(A)** $U + m$.
- **(B)** Um.
- **(C)** U^m.
- **(D)** m^U.
- **(E)** $\binom{U+m}{m}$.
- **(F)** The answer is blowing in the wind.

How many bits one need?

Let $\mathcal{H}$ be a set of functions from $\mathcal{U} = \{1, \ldots, U\}$ to $\{1, \ldots, m\}$. Specifying a function in $\mathcal{H}$ requires:

- **(A)** $O(U + m)$ bits.
- **(B)** $O(Um)$ bits.
- **(C)** $O(U^m)$ bits.
- **(D)** $O\left(m^U\right)$ bits.
- **(E)** $O(\log |\mathcal{H}|)$ bits.
- **(F)** Many many bits. At least two.

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3. Randomized guarantee: should have the property that for any *fixed* set $S \subseteq \mathcal{U}$ of size m the expected number of collisions for a function chosen from $\mathcal{H}$ should be "small". Here the expectation is over the randomness in choice of h.

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1. Yes... But what it means for $\mathcal{H}$ to be good and compact.

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3. Second property is stronger than the first and the crucial issue.

Definition

A family of hash function $\mathcal{H}$ is **(2-)universal** if for all distinct $x, y \in \mathcal{U}$, $\Pr_h[h(x) = h(y)] = 1/m$ where m is the table size.

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Note: The set of all hash functions satisfies stronger properties!

Analyzing Universal Hashing

1. T is hash table of size m.
2. $S \subseteq \mathcal{U}$ is a **fixed** set of size $\leq m$.
3. h is chosen randomly from a universal hash family $\mathcal{H}$.
4. x is a *fixed* element of $\mathcal{U}$.

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3. For $y \in S$ let A_y be the event that x, y collide and D_y be the corresponding indicator variable.

Analyzing Universal Hashing

Continued...

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$$\Rightarrow \mathrm{E}[\ell(x)] = \sum_{y \in S} \mathrm{E}[D_y] \qquad \text{linearity of expectation}$$

$$= \sum_{y \in S} Pr[h(x) = h(y)]$$

$$= \sum_{y \in S} \frac{1}{m} \qquad \text{(since } \mathcal{H} \text{ is a universal hash family)}$$

$$= |S|/m$$

$$= \frac{n}{m}$$

$$\leq 1 \qquad \text{(if } |S| \leq m\text{)}$$

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   [Lower bound holds even under stronger assumptions.]

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We need *compactly representable* universal family.

Compact Universal Hash Family

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

1. Choose a **prime** number $p > N$. Define function $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.

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Theorem

$\mathcal{H}$ *is a universal hash family.*

Compact Universal Hash Family

Parameters: $N = |\mathcal{U}|$, $m = |T|$, $n = |S|$

1. Choose a **prime** number $p > N$. Define function $h_{a,b}(x) = ((ax + b) \mod p) \mod m$.
2. Let $\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0\}$ ($\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$). Note that $|\mathcal{H}| = p(p-1)$.

Theorem

$\mathcal{H}$ is a universal hash family.

Comments:

1. $h_{a,b}$ can be evaluated in $O(1)$ time.
2. Easy to store, *i.e.*, just store a, b. Easy to sample.

Some math required...

Lemma (LemmaUnique)

Let p be a prime number, and $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$.
x: an integer number in $\mathbb{Z}_p$, $x \neq 0$
$\implies$ *There exists a unique $y \in \mathbb{Z}_p$ s.t. $xy = 1 \mod p$.*

In other words: For every element there is a unique inverse.
$\implies$ set $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ when working modulo p is a *field*.

Proof of LemmaUnique

Claim

Let p *be a prime number. For any* $x, y, z \in \{1, \ldots, p-1\}$ *s.t.* $y \neq z$, *we have that* $xy \mod p \neq xz \mod p$.

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Proof.

Assume for the sake of contradiction $xy \mod p = xz \mod p$. Then

$$
\begin{aligned}
& x(y-z) = 0 \mod p \\
\implies & \quad p \text{ divides } x(y-z) \\
\implies & \quad p \text{ divides } y-z \\
\implies & \quad y-z=0 \implies y=z
\end{aligned}
$$

And that is a contradiction. □

Proof of LemmaUnique

Lemma (LemmaUnique)

Let p be a prime number,
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$\implies$ *There exists a unique y s.t. $xy = 1 \mod p$.*

Proof.

By the above claim if $xy = 1 \mod p$ and $xz = 1 \mod p$ then $y = z$. Hence uniqueness follows.

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Existence. For any $x \in \{1, \ldots, p-1\}$ we have that
$\{x * 1 \mod p, x * 2 \mod p, \ldots, x * (p-1) \mod p\} =$

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Existence. For any $x \in \{1, \ldots, p-1\}$ we have that $\{x * 1 \mod p, x * 2 \mod p, \ldots, x * (p-1) \mod p\} = \{1, 2, \ldots, p-1\}$.
$\implies$ There exists a number $y \in \{1, \ldots, p-1\}$ such that $xy = 1 \mod p$. □

Proof of the Theorem: Outline

$h_{a,b}(x) = ((ax + b) \mod p) \mod m)$.

Theorem

$\mathcal{H} = \{h_{a,b} \mid a, b \in \mathbb{Z}_p, a \neq 0\}$ *is universal.*

Proof.

Fix $x, y \in \mathcal{U}$. We need to show that $\Pr_{h_{a,b} \sim \mathcal{H}}[h_{a,b}(x) = h_{a,b}(y)] \leq 1/m$. Note that $|\mathcal{H}| = p(p-1)$.

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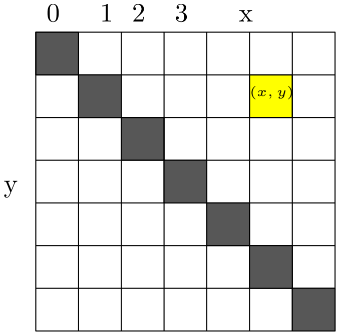
1. Let (a, b) (equivalently $h_{a,b}$) be *bad* for x, y if $h_{a,b}(x) = h_{a,b}(y)$.
2. **Claim:** Number of bad (a, b) is at most $p(p-1)/m$.
3. Total number of hash functions is $p(p-1)$ and hence probability of a collision is $\leq 1/m$. □

Intuition for the Claim

$g_{a,b}(x) = (ax + b) \ mod \ p, \ \ h_{a,b}(x) = (g_{a,b}(x)) \ mod \ m$

First map $x \neq y$ to $r = g_{a,b}(x)$ and $s = g_{a,b}(y)$.

LemmaUnique $\implies r \neq s$



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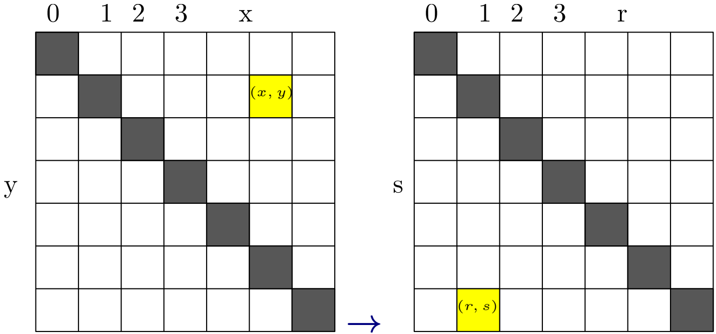

As (a, b) varies, (r, s) takes all possible $p(p-1)$ values. Since (a, b) is picked u.a.r., every value of (r, s) has equal probability.

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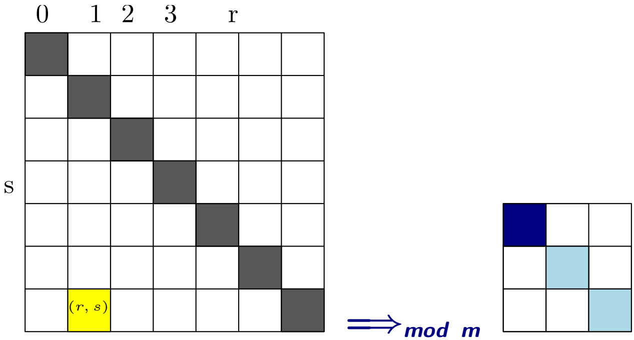

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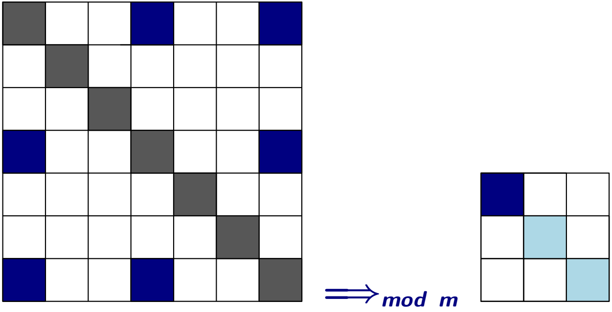

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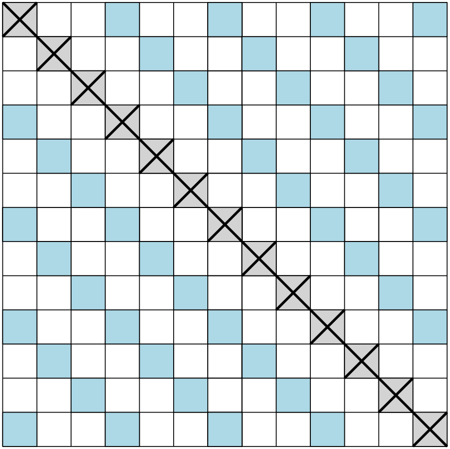
1. First part of mapping maps (x, y) to a random location $(g_{a,b}(x), g_{a,b}(y))$ in the "matrix".
2. $(g_{a,b}(x), g_{a,b}(y))$ is not on main diagonal.
3. All blue locations are "bad" – map by **mod** m to a location of collision.
4. But... at most $1/m$ fraction of allowable locations in the matrix are bad.



We need

to show at most 1/m fraction of bad $h_{a,b}$

$$h_{a,b}(x) = (((ax + b) \ mod \ p) \ modm)$$

2 lemmas ...

Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \ mod \ p$ and $s = (ay + b) \ mod \ p$.

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Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \ mod \ p$ and $s = (ay + b) \ mod \ p$.

1. 1-to-1 correspondence between $p(p - 1)$ pairs of (a, b) (equivalently $h_{a,b}$) and $p(p - 1)$ pairs of (r, s).

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Fix $x \neq y \in \mathbb{Z}_p$, and let $r = (ax + b) \; mod \; p$ and $s = (ay + b) \; mod \; p$.

1. 1-to-1 correspondence between $p(p-1)$ pairs of (a, b) (equivalently $h_{a,b}$) and $p(p-1)$ pairs of (r, s).
2. Out of all possible $p(p-1)$ pairs of (r, s), at most $p(p-1)/m$ fraction satisfies $r \; mod \; m = s \; mod \; m$.

Some Lemmas

Lemma

If $x \neq y$ then for any $a, b \in \mathbb{Z}_p$ such that $a \neq 0$, we have

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Suppose not

$$ax + b \mod p = ay + b \mod p \Rightarrow a(x - y) \mod p = 0$$

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Proof.

Suppose not

$$ax + b \mod p = ay + b \mod p \Rightarrow a(x - y) \mod p = 0$$

But, $a \neq 0$ and $(x - y) \neq 0$. And a and $(x - y)$ cannot divide p since p is prime and $a < p$ and $(x - y) < p$. Contradiction! □

Some Lemmas

Lemma

If $x \neq y$ *then for each* (r, s) *such that* $r \neq s$ *and* $0 \leq r, s \leq p - 1$ *there is exactly* **one** a, b *such that*

$$ax + b \mod p = r \text{ and } ay + b \mod p = s.$$

Proof.

Solve the two equations:

$$ax + b = r \mod p \qquad \text{and} \qquad ay + b = s \mod p$$

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Proof.

Solve the two equations:

$$ax + b = r \mod p \qquad \text{and} \qquad ay + b = s \mod p$$

We get $a = \frac{r-s}{x-y} \mod p$ and $b = r - ax \mod p$. □

One-to-one correspondence between (a, b) and (r, s)

Understanding the hashing

Once we fix a and b, and we are given a value x, we compute the hash value of x in two stages:

1. **Compute**: $r \leftarrow (ax + b) \mod p$.
2. **Fold**: $r' \leftarrow r \mod m$

Collision...

Given two distinct values x and y they might collide only because of folding.

Lemma

$\#$ not equal pairs (r, s) of $\mathbb{Z}_p \times \mathbb{Z}_p$ that are folded to the same number is $p(p-1)/m$.

Folding numbers

Lemma

 # pairs $(r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ *such that* $r \neq s$ *and* $r \mod m = s \mod m$ *(folded to the same number) is* $p(p-1)/m$*.*

Proof.

Consider a pair $(r, s) \in \{0, 1, \ldots, p-1\}^2$ s.t. $r \neq s$. Fix r:

1. $a = r \mod m$.

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Lemma

 # pairs $(r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ *such that* $r \neq s$ *and* $r \mod m = s \mod m$ *(folded to the same number) is* $p(p-1)/m$.

Proof.

Consider a pair $(r, s) \in \{0, 1, \ldots, p-1\}^2$ s.t. $r \neq s$. Fix r:

1. $a = r \mod m$.
2. There are $\lceil p/m \rceil$ values of s that fold into a. That is

$$r \mod m = s \mod m.$$

3. One of them is when $r = s$.
4. $\implies$ # of colliding pairs

Folding numbers

Lemma

$\#$ pairs $(r, s) \in \mathbb{Z}_p \times \mathbb{Z}_p$ such that $r \neq s$ and $r \mod m = s \mod m$ (folded to the same number) is $p(p-1)/m$.

Proof.

Consider a pair $(r, s) \in \{0, 1, \ldots, p-1\}^2$ s.t. $r \neq s$. Fix r:

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3. One of them is when $r = s$.
4. $\implies \#$ of colliding pairs $(\lceil p/m \rceil - 1)p \leq (p-1)p/m$

□

Proof of Claim

of bad pairs is $\mathbf{p(p-1)/m}$

Proof.

Let $a, b \in \mathbb{Z}_p$ such that $a \neq 0$ and $h_{a,b}(x) = h_{a,b}(y)$.

1. Let $r = ax + b \mod p$ and $s = ay + b \mod p$.
2. Collision if and only if $r \mod m = s \mod m$.
3. (Folding error): Number of pairs (r, s) such that $r \neq s$ and $0 \leq r, s \leq p - 1$ and $r \mod m = s \mod m$ is $p(p-1)/m$.
4. From previous lemma there is one-to-one correspondence between (a, b) and (r, s). Hence total number of bad (a, b) pairs is $p(p-1)/m$.

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□

Prob of x and y to collide: $\frac{\#\text{ bad } (a,b) \text{ pairs}}{\#(a,b) \text{ pairs}} = \frac{p(p-1)/m}{p(p-1)} = \frac{1}{m}$.

Rehashing, amortization and...

... making the hash table dynamic

So far we assumed fixed S of size $\simeq m$.

Question: What happens as items are inserted and deleted?

1. If $|S|$ grows to more than cm for some constant c then hash table performance clearly degrades.
2. If $|S|$ stays around $\simeq m$ but incurs many insertions and deletions then the initial random hash function is no longer random enough!

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Solution: Rebuild hash table periodically!

1. Choose a new table size based on current number of elements in table.
2. Choose a new random hash function and rehash the elements.
3. Discard old table and hash function.

Question: When to rebuild? How expensive?

Rebuilding the hash table

1. Start with table size m where m is some estimate of $|S|$ (can be some large constant).
2. If $|S|$ grows to more than twice current table size, build new hash table (choose a new random hash function) with double the current number of elements. Can also use similar trick if table size falls below quarter the size.
3. If $|S|$ stays roughly the same but more than $c|S|$ operations on table for some chosen constant c (say $\mathbf{10}$), rebuild.

The **amortize** cost of rebuilding to previously performed operations. Rebuilding ensures $O(\mathbf{1})$ expected analysis holds even when S changes. Hence $O(\mathbf{1})$ expected look up/insert/delete time *dynamic* data dictionary data structure!