Network Flow Algorithms

Lecture 14

Feb 8, 2018

Most slides are courtesy Prof. Chekuri

Question

Given a network $G = (V, E)$ with capacity $c(e)$ on edge e, let $f : E \rightarrow \mathbb{R}^+$ be a valid edge flow.

If there is an s-t path p such that on all edges of this path $f(e) < c(e)$. Then,

1. We can send some more flow from s to t.
2. f is not a maximum flow in G.
3. Both of the above
4. None of the first two.

Part I

Algorithm(s) for Maximum Flow

Recall...

Given a network $G = (V, E)$ with capacity non-negative $c(e)$ on each edge e, an s-t (edge-based) flow $f : E \to \mathbb{R}^+$ satisfies.

Capacity constraints: $f(e) \leq c(e)$ for all $e \in E$.

Flow conservation: For all vertices $v \in V$ other than s, t,

$$(\text{flow in to } v) = (\text{flow out of } v)$$

Flow value:

$$v(f) = (\text{flow out of } s) - (\text{flow in to } s)$$

The Maximum-Flow Problem

Problem

Input A network G with capacity c and source s and sink t.

Goal Find flow of **maximum** value from s to t.

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Exercise: Given G, s, t as above, show that one can remove all edges into s and all edges out of t without affecting the flow value between s and t.

Flow value: $v(f) = (\text{flow out of } s)$

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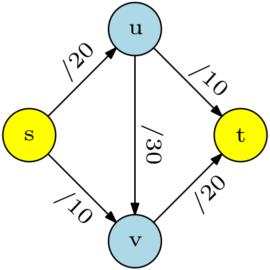
Greedy Approach



1. Begin with $f(e) = 0$ for each edge.
2. Find a s-t path P with $f(e) < c(e)$ for every edge $e \in P$.
3. **Augment** flow along this path.
4. Repeat augmentation for as long as possible.

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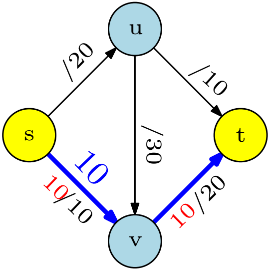

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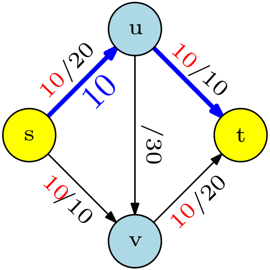

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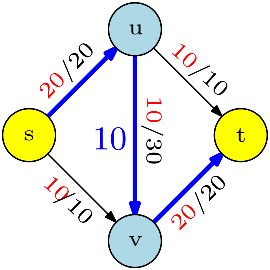

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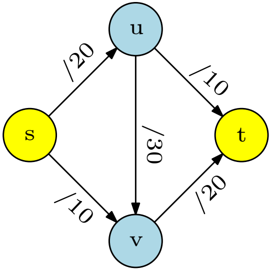

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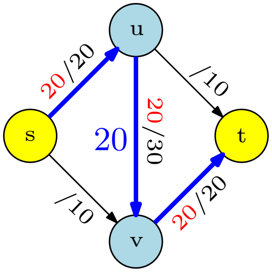

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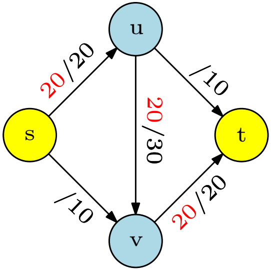

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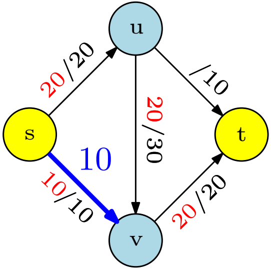

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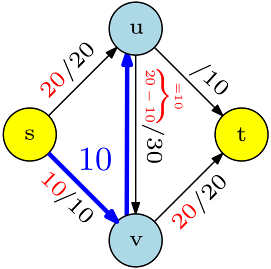

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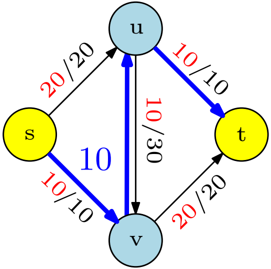

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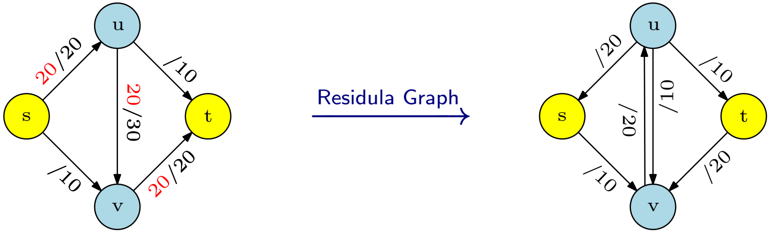

Residual graph has...

Given a network with n vertices and m edges, and a valid flow f in it, the residual network G_f, has

(A) m edges.

(B) $\leq 2m$ edges.

(C) $\leq 2m + n$ edges.

(D) $4m + 2n$ edges.

(E) nm edges.

(F) just the right number of edges - not too many, not too few.

Residual Graph Property

Observation: Residual graph captures the "residual" problem exactly.

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Definition of $+$ and - for flows is intuitive and the above lemmas are easy in some sense but a bit messy to formally prove.

Residual Graph Property – Intuition

Let f and f' be two flows in G with $v(f') \geq v(f)$.

Residual Graph Property: Implication

Recursive algorithm for finding a maximum flow:

```
MaxFlow(G, s, t):
    if the flow from s to t is 0 then
        return 0
    Find any flow f with v(f) > 0 in G
    Recursively compute a maximum flow f' in G_f
    Output the flow f + f'
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Iterative algorithm for finding a maximum flow:

```
MaxFlow(G, s, t):
    Start with flow f that is 0 on all edges
    while there is a flow f' in G_f with v(f') > 0 do
        f = f + f'
        Update G_f

    Output f
```

Ford-Fulkerson Algorithm

algFordFulkerson

```
for every edge e, f(e) = 0
G_f is residual graph of G with respect to f
while G_f has a simple s-t path do
    let P be simple s-t path in G_f
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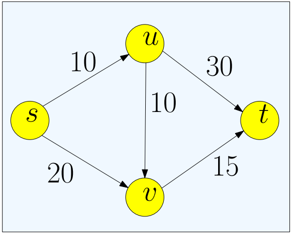
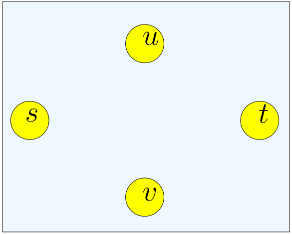
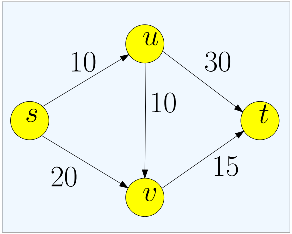
```
augment(f, P)
    let b be bottleneck capacity,
        i.e., min capacity of edges in P (in G_f)
    for each edge (u, v) in P do
        if e = (u, v) is a forward edge then
            f(e) = f(e) + b
        else (* (u, v) is a backward edge *)
            let e = (v, u) (* (v, u) is in G *)
            f(e) = f(e) − b
    return f
```

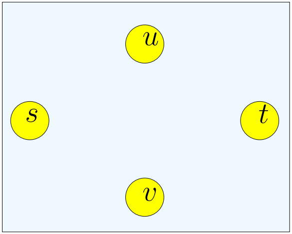
Example

f ***G_f***









Example continued

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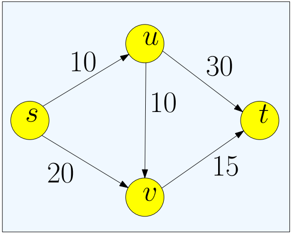

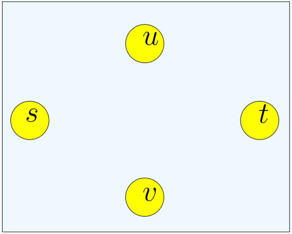

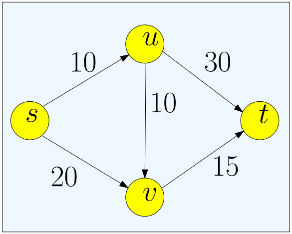

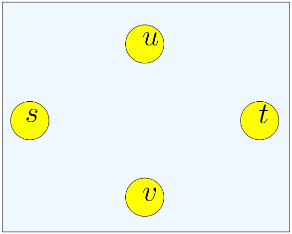

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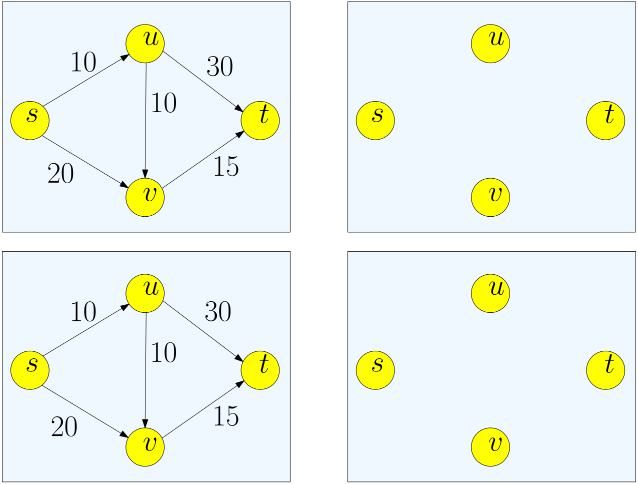

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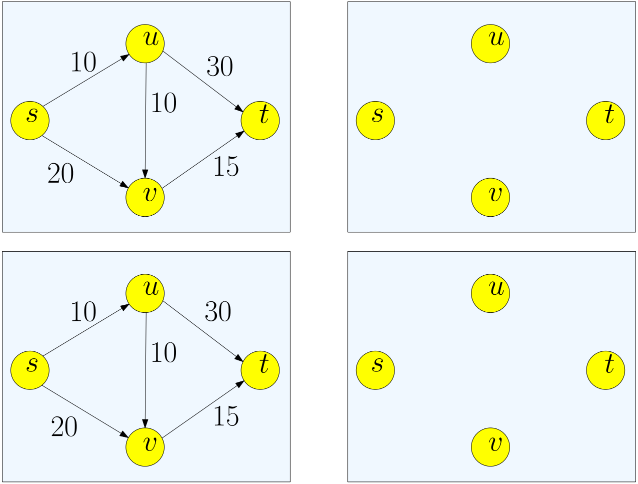

Properties about Augmentation: Flow

Lemma

If f is a flow and P is a simple s-t path in G_f, then $f' = \texttt{augment}(f, P)$ is also a flow.

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2. Conservation constraint: Let v be an internal node. Let e_1, e_2 be edges of P incident to v. Four cases based on whether e_1, e_2 are forward or backward edges. Check cases (see fig next slide). □

Properties of Augmentation

Conservation Constraint

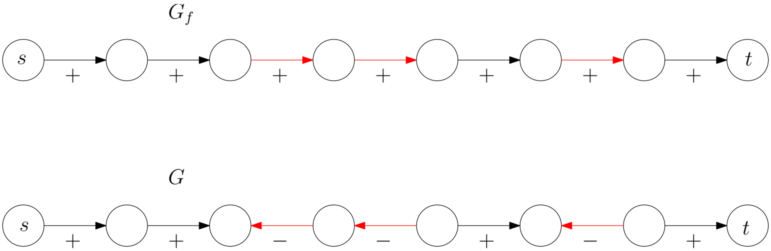

Figure: Augmenting path P in G_f and corresponding change of flow in G. Red edges are backward edges.

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Integer Flow

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And so flow after augmentation is an integer. □

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1. First edge e in P must leave s.

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1. First edge e in P must leave s.
2. Since no incoming edges to s in G, e is a forward edge.
3. P is simple and so never returns to s.
4. Thus, value of flow increases by the flow on edge e. □

Since edges in G_f have integer capacities, $v(f') \geq v(f) + 1$.

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Running time

1. Number of iterations $\leq C$.
2. Number of edges in $G_f \leq 2m$.
3. Time to find augmenting path is $O(n+m)$.
4. Running time is $O(C(n+m))$ (or $O(mC)$).

Efficiency of Ford-Fulkerson

Running time $= O(mC)$ is not polynomial. Can the running time be as $\Omega(mC)$ or is our analysis weak?

Ford-Fulkerson can take $\Omega(C)$ iterations.

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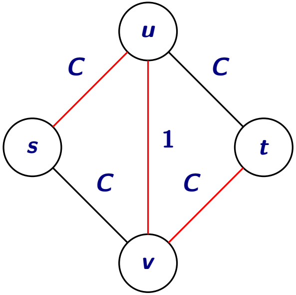

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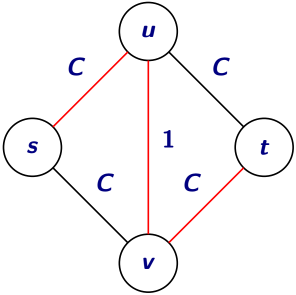

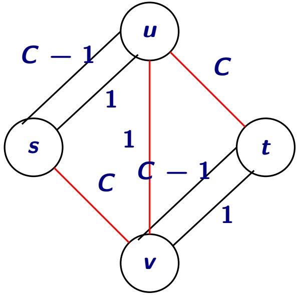

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Recalling Cuts

Definition

Given a flow network an **s-t cut** is a set of edges $E' \subset E$ such that removing E' *disconnects* s from t: in other words there is no directed $s \to t$ path in $E - E'$. **Capacity** of cut E' is $\sum_{e \in E'} c(e)$.

Let $A \subset V$ such that

1. $s \in A$, $t \notin A$, and
2. $B = V \setminus -A$ and hence $t \in B$.

Define $(A, B) = \{(u, v) \in E \mid u \in A, v \in B\}$

Claim

(A, B) *is an* s-t *cut.*

Recall: Every *minimal* s-t cut E' is a cut of the form (A, B).

Ford-Fulkerson Correctness

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If there is no s-t path in G_f then there is some cut (A, B) such that $v(f) = c(A, B)$

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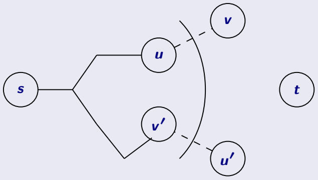

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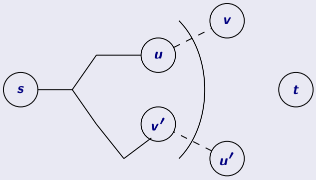
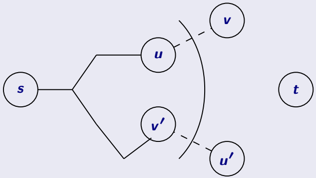


1. $s \in A$ and $t \in B$. So (A, B) is an s-t cut in G.
2. If $e = (u, v) \in G$ with $u \in A$ and $v \in B$, then $f(e) = c(e)$ (saturated edge) because otherwise v is reachable from s in G_f.

□

Lemma Proof Continued

Proof.

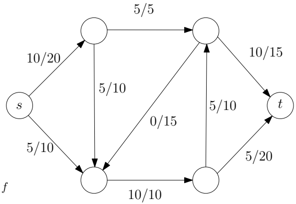


1. If $e = (u', v') \in G$ with $u' \in B$ and $v' \in A$, then $f(e) = 0$ because otherwise u' is reachable from s in G_f
2. Thus,

$$\begin{aligned} v(f) &= f^{\text{out}}(A) - f^{\text{in}}(A) \\ &= f^{\text{out}}(A) - 0 \\ &= c(A,B) - 0 \\ &= c(A,B). \end{aligned}$$

□

Example



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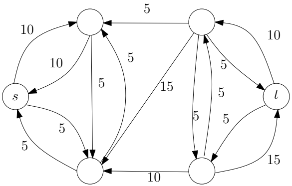

Residual graph G_f: no s-t path

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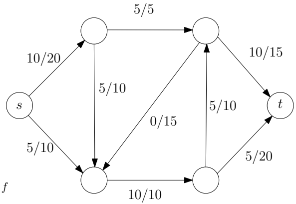

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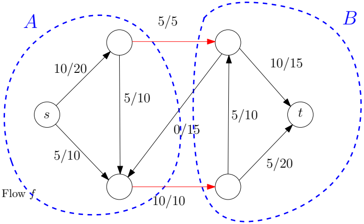

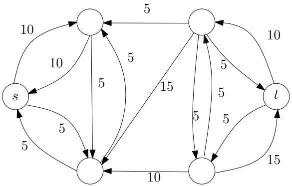

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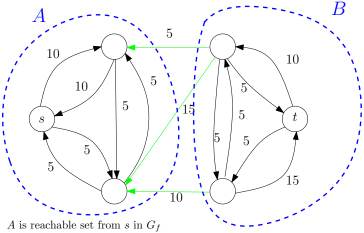

A is reachable set from s in G_f

Ford-Fulkerson Correctness

Theorem

The flow returned by the algorithm is the maximum flow.

Proof.

1. For any flow f and s-t cut (A, B), $v(f) \leq c(A, B)$.
2. For flow f^* returned by algorithm, $v(f^*) = c(A^*, B^*)$ for some s-t cut (A^*, B^*).
3. Hence, f^* is maximum.

□

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G, the value of a maximum s-t flow is equal to the capacity of the minimum s-t cut.

Proof.

Ford-Fulkerson algorithm terminates with a maximum flow of value equal to the capacity of a (minimum) cut. □

Max-Flow Min-Cut Theorem and Integrality of Flows

Theorem

For any network G with integer capacities, there is a maximum s-t flow that is integer valued.

Proof.

Ford-Fulkerson algorithm produces an integer valued flow when capacities are integers. □

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Running time is essentially the same as finding a maximum flow.

Note: Given G and a flow f there is a linear time algorithm to check if f is a maximum flow and if it is, outputs a minimum cut. How?

Does it terminate?

(A) **algFordFulkerson** always terminates.

(B) **algFordFulkerson** might not terminate if the input has real numbers.

(C) **algFordFulkerson** might not terminate if the input has rational numbers.

(D) **algFordFulkerson** might not terminate if the input is only integer numbers that are sufficiently large.

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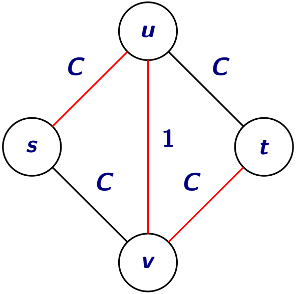

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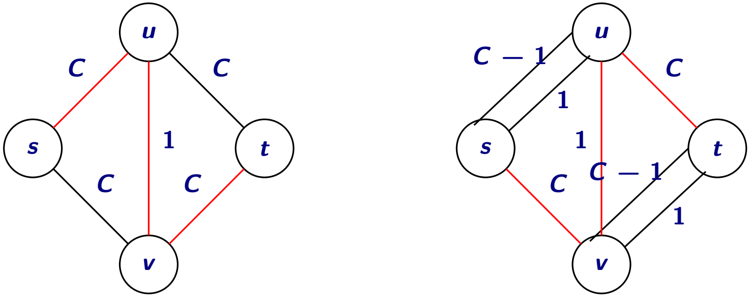

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Question: Is there a variant of Ford-Fulkerson that leads to a polynomial time algorithm? Can we choose an augmenting path in some clever way? Yes! Two variants.

1. Choose the augmenting path with largest bottleneck capacity.
2. Choose the shortest augmenting path.

Part II

Polynomial-time Augmenting Path Algorithms

Augmenting along high capacity paths

Definition

Given $G = (V, E)$ with edge capacities and a path P, the bottlneck capacity of P is smallest capacity among edges of P.

Algorithm: In each iteration of Ford-Fulkerson choose an augmenting path with largest bottleneck capacity.

Question: How many iterations does the algorithm take?

Finding path with largest bottleneck capacity

G_f - residual network with (residual) capacities.
n vertices and m edges.
Finding the s-t path with largest bottleneck capacity can be done (faster is better) in:

- **(A)** $O(n + m)$
- **(B)** $O(m + n \log n)$
- **(C)** $O(nm)$
- **(D)** $O(m^2)$
- **(E)** $O(m^3)$

time (expected or deterministic is fine here).

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    7. Can do binary search on the edge capacities. First, sort the edges by their capacities and then do binary search on that array as before.
    8. Algorithm's running time is $O(m \log m)$.
    9. Alternative algorithm: modify Dijkstra to get $O(m + n \log n)$.

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Hence, $\alpha_{i+1} - \alpha_i \geq \max\{1, (F^* - \alpha_i)/m\}$.

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This implies that algorithm terminates in $1 + m \ln F^*$ iterations. And $F^* \leq mC$ and hence algorithm terminates in $O(m \log mC)$ iterations.

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- Thus, $\alpha_{i+1} \geq \alpha_i + (F^* - \alpha_i)/m$.

Running time analysis

- Each iteration requires finding a max bottleneck capacity path in residual graph. Can be found in $O(n \log n + m)$ or in $O(m \log C)$ time.
- Number of iterations is $O(m \log mC)$.
- Hence overall running time is $O(m^2 \log mC \log C)$ or $O(mn \log n \log mC + m^2 \log mC)$.

Strongly polynomial time algorithm

Many problems has inputs with two types of information:

- combinatorial
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Graph problems: vertices and edges are combinatorial part and edge/vertex lengths/capacities are numerical.

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It is *pseudo-polynomial* if the run-time is polynomial assuming numerical data is in unary.

A strongly polynomial time algorithm for max flow

Algorithm: In each iteration of Ford-Fulkerson choose a shortest augmenting path in the residual graph.

algEdmondsKarp
  for every edge e, $f(e) = 0$
  G_f is residual graph of G with respect to f
  while G_f has a simple s-t path **do**
    Perform **BFS** in G_f
    P: shortest s-t path in G_f
    $f = \text{augment}(f, P)$
    Construct new residual graph G_f.

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Theorem

Algorithm terminates in $O(mn)$ iterations. Thus, overall running time is $O(m^2 n)$.

Orlin's Algorithm

- Currently, fastest strongly polynomial time algorithm runs in $O(mn)$ time.
- $O(mn)$ time is also sufficient to do flow-decomposition

You can state and use the above results in a black box fashion when using maximum flow algorithms in reductions.