CS 473: Algorithms

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Spring 2018

LP Duality

Lecture 20

April 3, 2018

Some of the slides are courtesy Prof. Chekuri1

Part I

Recall

Feasible Solutions and Lower Bounds

Consider the program

$$
\begin{array}{lrrl}
\text{maximize} & 4x_1 + & 2x_2 & \\
\text{subject to} & x_1 + & 3x_2 & \leq 5 \\
 & 2x_1 - & 4x_2 & \leq 10 \\
 & x_1 + & x_2 & \leq 7 \\
 & x_1 & & \leq 5
\end{array}
$$

Feasible Solutions and Lower Bounds

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Feasible Solutions and Lower Bounds

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1. $(2, 0)$ also feasible, and gives a better bound of 8.
2. How good is 8 when compared with σ^*?

Obtaining Upper Bounds

1. Let us multiply the first constraint by **2** and the and add it to second constraint

$$\begin{array}{rrrl} 2(& x_1 + & 3x_2 &) \leq 2(5) \\ +1(& 2x_1 - & 4x_2 &) \leq 1(10) \\ \hline & 4x_1 + & 2x_2 & \leq 20 \end{array}$$

2. Thus, 20 is an upper bound on the optimum value!

Generalizing ...

1. Multiply first equation by y_1, second by y_2, third by y_3 and fourth by y_4 (all of y_1, y_2, y_3, y_4 being positive) and add

$$\begin{array}{rrrl} y_1(& x_1 + & 3x_2 &) \leq y_1(5) \\ +y_2(& 2x_1 - & 4x_2 &) \leq y_2(10) \\ +y_3(& x_1 + & x_2 &) \leq y_3(7) \\ +y_4(& x_1 & &) \leq y_4(5) \\ \hline \multicolumn{4}{l}{(y_1 + 2y_2 + y_3 + y_4)x_1 + (3y_1 - 4y_2 + y_3)x_2 \leq \ldots} \end{array}$$

2. $5y_1 + 10y_2 + 7y_3 + 5y_4$ is an upper bound, provided coefficients of x_i are same as in the objective function, i.e.,

$$y_1 + 2y_2 + y_3 + y_4 = 4 \quad 3y_1 - 4y_2 + y_3 = 2$$

3. The best upper bound is when $5y_1 + 10y_2 + 7y_3 + 5y_4$ is minimized!

Dual LP: Example

Thus, the optimum value of program

$$\begin{aligned} \text{maximize} \quad & 4x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 3x_2 \leq 5 \\ & 2x_1 - 4x_2 \leq 10 \\ & x_1 + x_2 \leq 7 \\ & x_1 \leq 5 \end{aligned}$$

is upper bounded by the optimal value of the program

$$\begin{aligned} \text{minimize} \quad & 5y_1 + 10y_2 + 7y_3 + 5y_4 \\ \text{subject to} \quad & y_1 + 2y_2 + y_3 + y_4 = 4 \\ & 3y_1 - 4y_2 + y_3 = 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

Dual Linear Program

Given a linear program Π in canonical form

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad i = 1, 2, \ldots n \end{array}$$

the dual $\mathrm{Dual}(\Pi)$ is given by

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{n} b_i y_i & \\ \text{subject to} & \sum_{i=1}^{n} y_i a_{ij} = c_j & j = 1, 2, \ldots d \\ & y_i \geq 0 & i = 1, 2, \ldots n \end{array}$$

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Proposition

$\mathrm{Dual}(\mathrm{Dual}(\Pi))$ *is equivalent to* Π

Duality Theorems

Theorem (Weak Duality)

If x' *is a feasible solution to* Π *and* y' *is a feasible solution to* $\mathrm{Dual}(\Pi)$ *then* $c \cdot x' \leq y' \cdot b$.

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Theorem (Strong Duality)

If x^* *is an optimal solution to* Π *and* y^* *is an optimal solution to* $\mathrm{Dual}(\Pi)$ *then* $c \cdot x^* = y^* \cdot b$.

Many applications! Maxflow-Mincut theorem can be deduced from duality.

Weak Duality

Theorem (Weak Duality)

If x *is a feasible solution to* Π *and* y *is a feasible solution to* $\mathrm{Dual}(\Pi)$ *then* $c \cdot x \leq y \cdot b$.

We already saw the proof by the way we derived it but we will do it again formally.

Proof.

Since y' is feasible in $\mathrm{Dual}(\Pi)$: $y'A = c$

Weak Duality

Theorem (Weak Duality)

If $\mathbf{x}$ is a feasible solution to $\mathbf{\Pi}$ and $\mathbf{y}$ is a feasible solution to $\mathbf{Dual(\Pi)}$ then $\mathbf{c} \cdot \mathbf{x} \leq \mathbf{y} \cdot \mathbf{b}$.

We already saw the proof by the way we derived it but we will do it again formally.

Proof.

Since $\mathbf{y'}$ is feasible in $\mathbf{Dual(\Pi)}$: $\mathbf{y'A = c}$

Therefore $\mathbf{c \cdot x' = y'Ax'}$

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Proof.

Since y' is feasible in $\mathrm{Dual}(\Pi)$: $y'A = c$

Therefore $c \cdot x' = y'Ax'$

Since x' is feasible in Π, $Ax' \leq b$ and hence,

$$c \cdot x' = y'Ax' \leq y' \cdot b$$

Strong Duality ≡ Complementary Slackness

$$\begin{array}{ll} \textbf{maximize}: & c \cdot x \\ \text{subject to} & Ax \leq b \end{array} \quad \xrightarrow{\textit{Dual}} \quad \begin{array}{ll} \textbf{minimize}: & y \cdot b \\ \text{subject to} & yA = c \\ & y \geq 0 \end{array}$$

Definition (Complementary Slackness)

x feasible in Π and y feasible in $\mathrm{Dual}(\Pi)$, s.t.,

$$\forall i = 1..n, \quad y_i > 0 \;\Rightarrow\; (Ax)_i = b_i$$

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Q: Why complementary slackness is satisfied at the optimum?

Complementary Slackness: Geometric View

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$\boldsymbol{x}^*$ **:** Optimum vertex. First $\boldsymbol{d}$ inequalities tight at $\boldsymbol{x}^*$.

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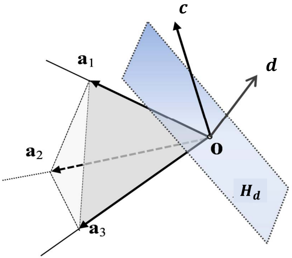

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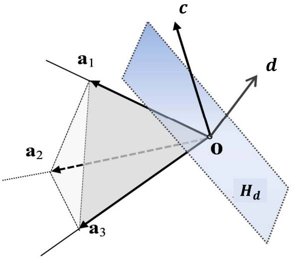

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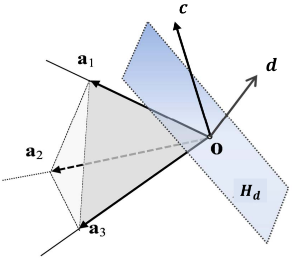

$\Rightarrow$ There exists a hyperplane separating c from the cone.
Suppose cone is on the negative side, and c on the positive size. If the d is the normal vector of the hyperplane, then formally,

$$\hat{A}d < 0, \qquad c \cdot d > 0$$

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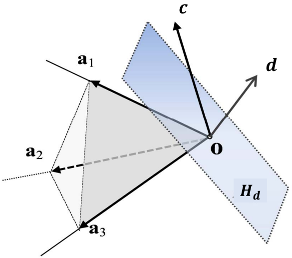

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Choose v. v. tiny $\epsilon > 0$ such that $\tilde{A}(x^* + \epsilon d) \leq \tilde{b}$.

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$$\boldsymbol{c \cdot (x^* + \epsilon d) = c \cdot x^* + \epsilon (c \cdot d)}$$

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$\boldsymbol{c \cdot (x^* + \epsilon d) = c \cdot x^* + \epsilon(c \cdot d) > c \cdot x^*} \Rightarrow$

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$\boldsymbol{c} \cdot (\boldsymbol{x}^* + \epsilon \boldsymbol{d}) = \boldsymbol{c} \cdot \boldsymbol{x}^* + \epsilon(\boldsymbol{c} \cdot \boldsymbol{d}) > \boldsymbol{c} \cdot \boldsymbol{x}^* \Rightarrow \boldsymbol{x}^*$ **is NOT optimum!**

Proof of Strong Duality

x^* **:** Optimum vertex. First d inequalities tight at x^*.

$$Ax^* \leq b \quad \text{splits into} \quad \hat{A}x^* = \hat{b}, \qquad \tilde{A}x^* < \tilde{b}$$

Then c IS in the cone of rows of $\hat{A}$.

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$\Leftrightarrow$ $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ satisfies strong duality, $\boldsymbol{c} \cdot \boldsymbol{x}^* = \boldsymbol{y}^* \cdot \boldsymbol{b}$. (implies $\boldsymbol{y}^*$ optimal in $\mathbf{Dual}(\boldsymbol{\Pi})$).

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Optimization vs Feasibility

Suppose we want to solve LP of the form:

$$\max \boldsymbol{c}\boldsymbol{x} \text{ subject to } \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$

It is an optimization problem. Can we reduce it to a decision problem?

Optimization vs Feasibility

Suppose we want to solve LP of the form:

$$\textbf{max } \boldsymbol{cx} \text{ subject to } \boldsymbol{Ax} \leq \boldsymbol{b}$$

It is an optimization problem. Can we reduce it to a decision problem? Yes, via binary search. Find the largest values of $\boldsymbol{\sigma}$ such that the system of inequalities

$$\boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{cx} \geq \boldsymbol{\sigma}$$

is *feasible*. Feasible implies that there is at least one solution.

Optimization vs Feasibility

Suppose we want to solve LP of the form:

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$$Ax \leq b, cx \geq \sigma$$

is *feasible*. Feasible implies that there is at least one solution. Caveat: to do binary search need to know the range of numbers. Skip for now since we need to worry about precision issues etc.

Certificate for (in)feasibility

Suppose we have a system of m inequalities in n variables defined by

$$Ax \leq b$$

- How can we convince some one that there is a feasible solution?
- How can we convince some one that there is **no** feasible solution?

Theorem of the Alternatives

Theorem

Let $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^{m}$*. Exactly one of the following two holds*
(*i*) *The system* $Ax \leq b$ *is feasible.*
(*ii*) *There is a* $y \in \mathbb{R}^{m}$ *such that* $y \geq \mathbf{0}$ *and* $yA = \mathbf{0}$ *and* $yb < \mathbf{0}$*.*

In other words, if $Ax \leq b$ is infeasible we can demonstrate it as follows: Find a non-negative combination of the rows of A (given by certificate y) to get a contradiction $\mathbf{0} < \mathbf{0}$.

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$\mathbf{0} = (yA)x = y(Ax) =$

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In other words, if $\boldsymbol{Ax} \leq \boldsymbol{b}$ is infeasible we can demonstrate it as follows: Find a non-negative combination of the rows of $\boldsymbol{A}$ (given by certificate $\boldsymbol{y}$) to get a contradiction $\mathbf{0} < \mathbf{0}$.
$\mathbf{0} = (yA)x = y(Ax) = y \cdot b < \mathbf{0}$!

Theorem of the Alternatives

Theorem

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Exactly one of the following two holds

(i) *The system $Ax \leq b$ is feasible.*

(ii) *There is a $y \in \mathbb{R}^m$ such that $y \geq 0$ and $yA = 0$ and $yb < 0$.*

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The preceding theorem can also be used to prove strong duality.

Farkas' lemma is another such theorem. These are related.

Duality for another canonical form

Compactly, for the primal LP Π

$$\begin{array}{ll} \max & c \cdot x \\ \text{subject to} & Ax \leq b, \; x \geq 0 \end{array}$$

the dual LP is $\mathrm{Dual}(\Pi)$

$$\begin{array}{ll} \min & y \cdot b \\ \text{subject to} & yA \geq c, \; y \geq 0 \end{array}$$

Definition (Complementary Slackness)

x feasible in Π and y feasible in $\mathrm{Dual}(\Pi)$, s.t.,

$$\forall i = 1, \ldots, n, \quad y_i > 0 \; \Rightarrow \; (Ax)_i = b_i$$
$$\forall j = 1, \ldots, d, \quad x_j > 0 \; \Rightarrow \; (yA)_j = c_j$$

In General...

from Jeff's notes

Primal	Dual
$\max c \cdot x$	$\min y \cdot b$
$\sum_j a_{ij}x_j \le b_i$	$y_i \ge 0$
$\sum_j a_{ij}x_j \ge b_i$	$y_i \le 0$
$\sum_j a_{ij}x_j = b_i$	—
$x_j \ge 0$	$\sum_i y_i a_{ij} \ge c_j$
$x_j \le 0$	$\sum_i y_i a_{ij} \le c_j$
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Figure H.4. Constructing the dual of an arbitrary linear program.

Some Useful Duality Properties

Assume primal LP is a maximization LP.

- For a given LP, Dual is another LP. The variables in the dual correspond to "non-trival" primal constraints and vice-versa.
- Dual of the dual LP give us back the primal LP.
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- Primal and dual optimum solutions satisfy complementary slackness conditions (discussed soon).

Part II

Examples of Duality

Network flow

s-t flow in directed graph $G = (V, E)$ with capacities c. Assume for simplicity that no incoming edges into s.

$$\max \quad \sum_{(s,v)\in E} x(s,v)$$

$$\sum_{(u,v)\in E} x(u,v) - \sum_{(v,w)\in E} x(v,w) = 0 \quad \forall v \in V \setminus \{s,t\}$$

$$x(u,v) \le c(u,v) \quad \forall (u,v) \in E$$

$$x(u,v) \ge 0 \quad \forall (u,v) \in E.$$

Network flow: Equivalent formulation

Directed graph $G = (V, E)$, with capacities on edges. Add a t to s edge with infinite capacity. Now maximize flow on this edge.

$$\begin{aligned}
\textbf{max} \quad & x(t, s) \\
& \sum_{(u,v)\in E} x(u, v) - \sum_{(v,w)\in E} x(v, w) = 0 && \forall v \in V \\
& x(u, v) \le c(u, v) && \forall (u, v) \in E \setminus (t, s) \\
& x(u, v) \ge 0 && \forall (u, v) \in E.
\end{aligned}$$

Dual of Network Flow

Part III

Integer Linear Programming

Integer Linear Programming

Problem

Find a vector $x \in Z^d$ (integer values) that

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for } i = 1 \ldots n \end{array}$$

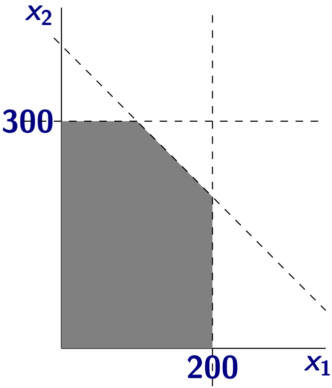
Input is matrix $A = (a_{ij}) \in \mathbb{R}^{n \times d}$, column vector $b = (b_i) \in \mathbb{R}^n$, and row vector $c = (c_j) \in \mathbb{R}^d$

Factory Example

$$\begin{array}{ll} \text{maximize} & x_1 + 6x_2 \\ \text{subject to} & x_1 \leq 200 \quad x_2 \leq 300 \quad x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{array}$$

Suppose we want x_1, x_2 to be integer valued.

Factory Example Figure



1. Feasible values of x_1 and x_2 are integer points in shaded region
2. Optimization function is a line; moving the line until it just leaves the final integer point in feasible region, gives optimal values

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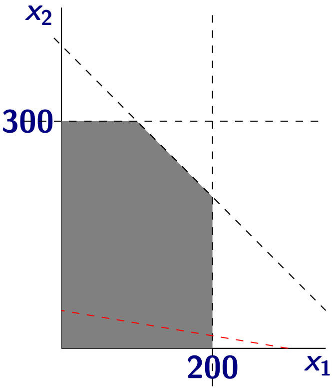

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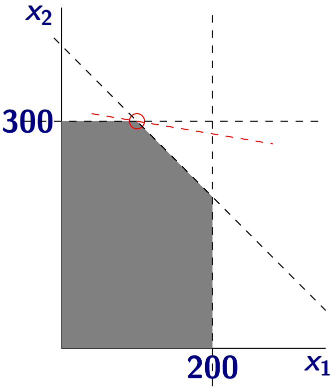

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Practice: integer programs are solved by a variety of methods

1. branch and bound
2. branch and cut
3. adding cutting planes
4. linear programming plays a fundamental role

Example: Maximum Independent Set

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in S. That is, if $u, v \in S$ then $(u, v) \notin E$.

Input Graph $G = (V, E)$

Goal Find maximum sized independent set in G

Example: Dominating Set

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is a dominating set if for all $v \in V$, either $v \in S$ or a neighbor of v is in S.

Input Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal Find minimum weight dominating set in G

Example: s-t minimum cut and implicit constraints

Input Graph $G = (V, E)$, edge capacities $c(e), e \in E$.
$s, t \in V$

Goal Find minimum capacity s-t cut in G.

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Luck or *Structure*:

1. Linear program for flows with integer capacities have integer vertices
2. Linear program for matchings in bipartite graphs have integer vertices
3. A complicated linear program for matchings in general graphs have integer vertices.

All of above problems can hence be solved efficiently.

Linear Programs with Integer Vertices

Meta Theorem: A combinatorial optimization problem can be solved efficiently if and only if there is a linear program for problem with integer vertices.

Consequence of the Ellipsoid method for solving linear programming.

In a sense linear programming and other geometric generalizations such as convex programming are the most general problems that we can solve efficiently.

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5. Integer Programming in **NP-Complete**. LP-based techniques critical in heuristically solving integer programs.