ILP and Reductions

Lecture 21

April 10, 2018

Most slides are courtesy Prof. Chekuri

Part I

Integer Linear Programming (ILP)

Integer Linear Programming

Problem

Find a vector $x \in Z^d$ **(integer values)** that

$$\begin{array}{ll} \text{maximize} & \sum_{j=1}^{d} c_j x_j \\ \text{subject to} & \sum_{j=1}^{d} a_{ij} x_j \leq b_i \quad \text{for } i = 1 \ldots n \end{array}$$

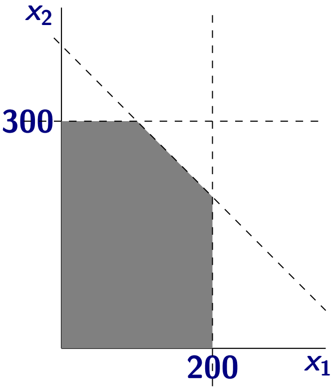
Input is matrix $A = (a_{ij}) \in \mathbb{R}^{n \times d}$, column vector $b = (b_i) \in \mathbb{R}^n$, and row vector $c = (c_j) \in \mathbb{R}^d$

Factory Example

$$\begin{array}{ll} \text{maximize} & x_1 + 6x_2 \\ \text{subject to} & x_1 \leq 200 \quad x_2 \leq 300 \quad x_1 + x_2 \leq 400 \\ & x_1, x_2 \geq 0 \end{array}$$

Suppose we want x_1, x_2 to be integer valued.

Factory Example Figure



1. Feasible values of x_1 and x_2 are **integer points in shaded region**
2. Optimization function is a line; moving the line until it just leaves the final integer point in feasible region, gives optimal values

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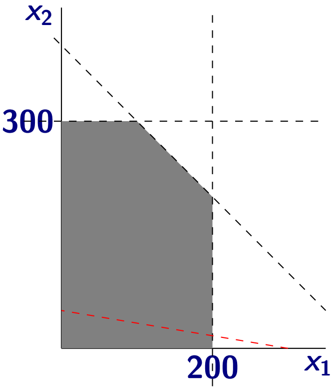

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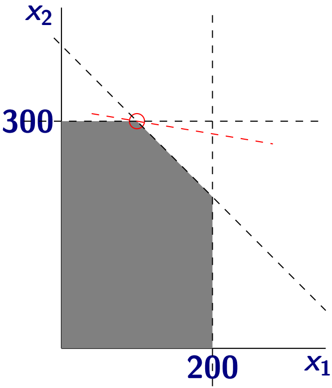

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Practice: integer programs are solved by a variety of methods

1. branch and bound
2. branch and cut
3. adding cutting planes
4. linear programming plays a fundamental role

Example: Maximum Independent Set

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is an independent set (also called a stable set) if for there are no edges between nodes in S. That is, if $u, v \in S$ then $(u, v) \notin E$.

Input Graph $G = (V, E)$

Goal Find maximum sized independent set in G

Example: Dominating Set

Definition

Given undirected graph $G = (V, E)$ a subset of nodes $S \subseteq V$ is a dominating set if for all $v \in V$, either $v \in S$ or a neighbor of v is in S.

Input Graph $G = (V, E)$, weights $w(v) \geq 0$ for $v \in V$

Goal Find minimum weight dominating set in G

Example: s-t minimum cut and implicit constraints

Input Graph $G = (V, E)$, edge capacities $c(e), e \in E$. $s, t \in V$

Goal Find minimum capacity s-t cut in G.

Linear Programs with Integer Vertices

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Luck or *Structure*:

1. Linear program for flows with integer capacities have integer vertices
2. Linear program for matchings in bipartite graphs have integer vertices
3. A complicated linear program for matchings in general graphs have integer vertices.

All of above problems can hence be solved efficiently.

Linear Programs with Integer Vertices

Meta Theorem: A combinatorial optimization problem can be solved efficiently if and only if there is a linear program for problem with integer vertices.

Consequence of the Ellipsoid method for solving linear programming.

In a sense linear programming and other geometric generalizations such as convex programming are the most general problems that we can solve efficiently.

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4. Duality is a critical tool in the theory of linear programming. Duality implies the Linear Programming is in **co-NP**. Do you see why?
5. Integer Programming in **NP-Complete**. LP-based techniques critical in heuristically solving integer programs.

Part II

Reductions

Reductions

A reduction from Problem X to Problem Y means (informally) that if we have an algorithm for Problem Y, we can use it to find an algorithm for Problem X.

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Using Reductions

1. We use reductions to find algorithms to solve problems.
2. We also use reductions to show that we can't find algorithms for some problems. (We say that these problems are hard.)

Example 1: Bipartite Matching and Flows

How do we solve the **Bipartite Matching** Problem?

Given a bipartite graph $G = (U \cup V, E)$ and number k, does G have a matching of size $\geq k$?

Solution

Reduce it to **Max-Flow**. G has a matching of size $\geq k$ iff there is a flow from s to t of value $\geq k$ in the auxiliary graph G'.

Types of Problems

Decision, Search, and Optimization

1. **Decision problem**. Example: given n, is n prime?.
2. **Search problem**. Example: given n, find a factor of n if it exists.
3. **Optimization problem**. Example: find the smallest prime factor of n.

Optimization and Decision problems

For max flow...

Problem (Max-Flow optimization version)

Given an instance G of network flow, find the maximum flow between s and t.

Problem (Max-Flow decision version)

Given an instance G of network flow and a parameter K, is there a flow in G, from s to t, of value at least K?

While using reductions and comparing problems, we typically work with the decision versions. Decision problems have Yes/No answers. This makes them easy to work with.

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2. The size of an instance I is the number of bits in its representation.
3. For an instance I, $sol(I)$ is a set of feasible solutions to I.
4. For optimization problems each solution $s \in sol(I)$ has an associated value.

Examples

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An instance of **Bipartite Matching** is a bipartite graph, and an integer k. The solution to this instance is “YES” if the graph has a matching of size $\geq k$, and “NO” otherwise.

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An instance of **Max-Flow** is a graph G with edge-capacities, two vertices s, t, and an integer k. The solution to this instance is "YES" if there is a flow from s to t of value $\geq k$, else 'NO".

What is an algorithm for a decision Problem X?

It takes as input an instance of X, and outputs either "YES" or "NO".

Using reductions to solve problems

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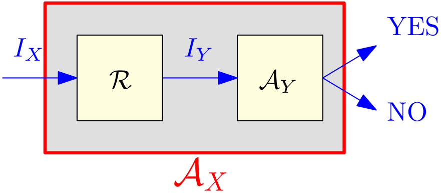
3. $\implies$ New algorithm for X:

$\mathcal{A}_X(I_X)$:
    // I_X: instance of X.
    $I_Y \leftarrow \mathcal{R}(I_X)$
    return $\mathcal{A}_Y(I_Y)$



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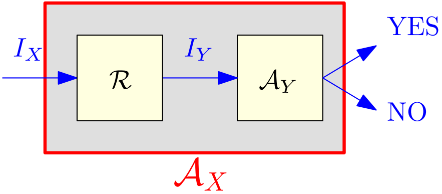

If $\mathcal{R}$ and $\mathcal{A}_Y$ polynomial-time $\implies$ $\mathcal{A}_X$ polynomial-time.

Comparing hardness of problems

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2. **Bipartite Matching** $\leq$ **Max-Flow**.
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4. $X \leq Y$:
   1. X is no harder than Y, or
   2. Y is at least as hard as X.

Polynomial-time Reductions

Efficient Algorithm: runs in polynomial-time.

To find efficient algorithms for problems, only polynomial-time reductions are useful. Reductions that take longer are not useful.

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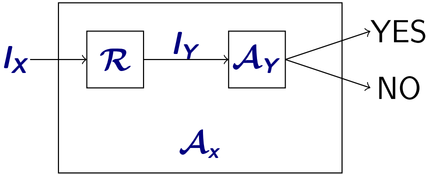
Then, polynomial-time algorithm $\mathcal{A}_Y$ for Y, gives an efficient algorithm for X.



Polynomial-time Reduction

A polynomial time reduction from a *decision* problem X to a *decision* problem Y is an *algorithm* $\mathcal{A}$ that has the following properties:

1. given an instance I_X of X, $\mathcal{A}$ produces an instance I_Y of Y
2. $\mathcal{A}$ runs in time polynomial in $|I_X|$.
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Proposition

If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

Such a reduction is called a **Karp reduction**. Most reductions we will need are Karp reductions.

Polynomial-time reductions and instance sizes

Proposition

Let $\mathcal{A}$ be a polynomial-time algorithm reducing X to Y. Then for any instance I_X of X, the size of the instance I_Y of Y produced from I_X by $\mathcal{A}$ is polynomial in the size of I_X.

Proof.

$\mathcal{A}$ is a polynomial-time algorithm and hence on input I_X of size $|I_X|$ it runs in time $p(|I_X|)$ for some polynomial $p()$.

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Note: Converse is not true. A reduction need not be polynomial-time even if output of reduction is of size polynomial in its input.

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2. $\mathcal{A}$ runs in time polynomial in $|I_X|$. **This implies that $|I_Y|$ (size of I_Y) is polynomial in $|I_X|$.**
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If $X \leq_P Y$ then a polynomial time algorithm for Y implies a polynomial time algorithm for X.

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Reductions again...

Let X and Y be two decision problems, such that X can be solved in polynomial time, and $X \leq_P Y$. Then

(A) Y can be solved in polynomial time.

(B) Y can NOT be solved in polynomial time.

(C) If Y is hard then X is also hard.

(D) None of the above.

(E) All of the above.

Transitivity of Reductions

Proposition

$X \leq_P Y$ and $Y \leq_P Z$ implies that $X \leq_P Z$.

Note: $X \leq_P Y$ does not imply that $Y \leq_P X$ and hence it is very important to know the FROM and TO in a reduction.

To prove $X \leq_P Z$ you need to show a reduction FROM X TO Z
In other words show that an algorithm for Z implies an algorithm for X.

Polynomial-time reductions and hardness

For decision problems X and Y, if $X \leq_P Y$, and Y has an efficient algorithm, X has an efficient algorithm.

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If you believe that **Independent Set** does not have an efficient algorithm, then can **Clique** have an efficient algorithm?

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If **Clique** had an efficient algorithm, so would **Independent Set**!

So, NO!

Using Reductions to show Hardness

We say that a problem is “hard” if there is no polynomial-time algorithm known for it (and it is believed that such an algorithm does not exist).

To show that Y is a hard problem:

- Start with an existing “hard” problem X
- Prove that $X \leq_P Y$
- Then we have shown that Y is a “hard” problem

Examples of hard problems

Problems

1. **SAT**
2. **3SAT**
3. **Independent Set** and **Clique**
4. **Vertex Cover**
5. **Set Cover**
6. **Hamilton Cycle**
7. **Knapsack** and **Subset Sum** and **Partition**
8. **Integer Programming**
9. ...

Part III

Examples of Reductions

Independent Sets and Cliques

Given a graph G, a set of vertices V' is:

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2. **clique**: *every* pair of vertices in V' is connected by an edge of G.

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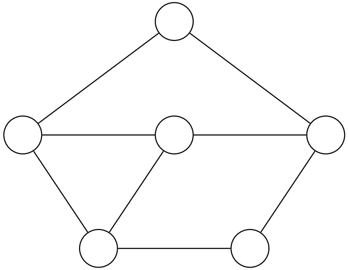

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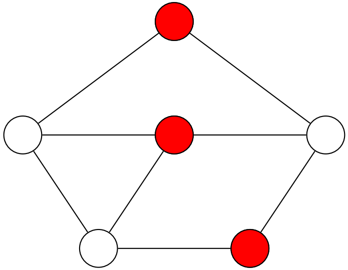

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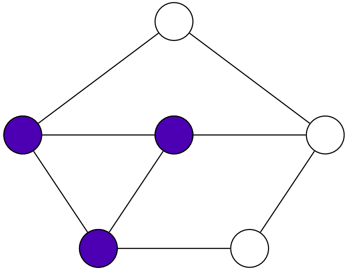

The **Independent Set** and **Clique** Problems

Problem: Independent Set

Instance: A graph G and an integer k.
Question: Does G has an independent set of size $\geq k$?

Problem: Clique

Instance: A graph G and an integer k.
Question: Does G has a clique of size $\geq k$?

Reducing **Independent Set** to **Clique**

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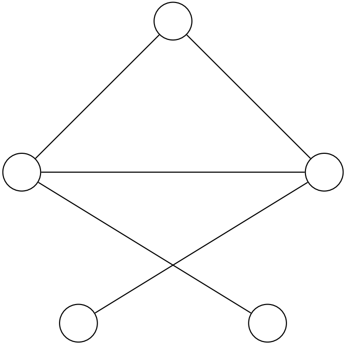

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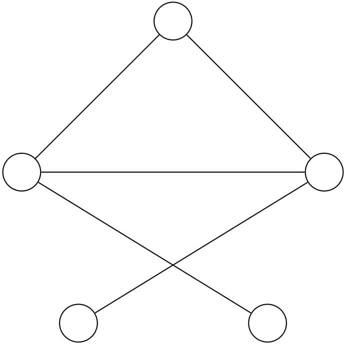

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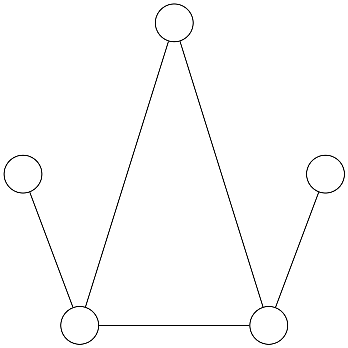

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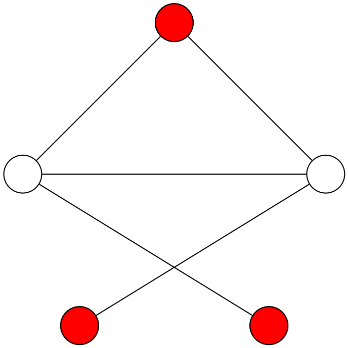

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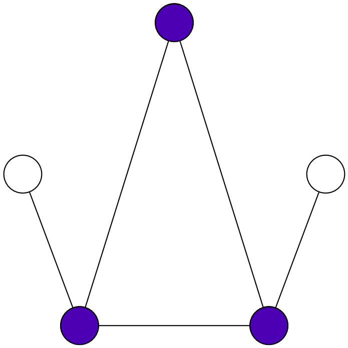

Independent Set and Clique

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   What does this mean?
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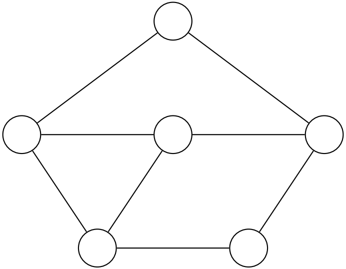

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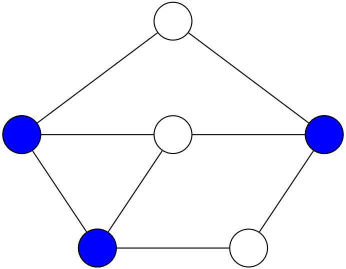
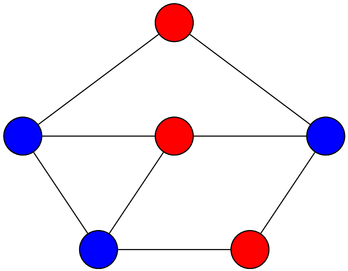


Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

1. A **vertex cover** if every $e \in E$ has at least one endpoint in S.



The Vertex Cover Problem

Problem (Vertex Cover)

Input: *A graph* G *and integer* k.
Goal: *Is there a vertex cover of size* $\leq k$ *in* G*?*

Can we relate **Independent Set** and **Vertex Cover**?

Relationship between...

Vertex Cover and Independent Set

Proposition

Let $G = (V, E)$ be a graph. S is an independent set if and only if $V \setminus S$ is a vertex cover.

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($\Rightarrow$) Let S be an independent set

1. Consider any edge $(u, v) \in E$. Is u or v in $V \setminus S$?

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3. Thus, either $u \in V \setminus S$ or $v \in V \setminus S$.
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($\Leftarrow$) Let $V \setminus S$ be some vertex cover:

1. Consider $u, v \in S$. Is $(u, v) \in E$?
2. $(u, v) \notin E$, as otherwise $V \setminus S$ does not cover (u, v).
3. $\implies$ S is thus an independent set. □

Independent Set $\leq_P$ Vertex Cover

1. G: graph with n vertices, and an integer k be an instance of the **Independent Set** problem.
2. **Claim.** G has an independent set of size $\geq k$ iff G has a vertex cover of size $\leq n - k$

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4. Therefore,
   Independent Set $\leq_P$ **Vertex Cover**.
   Also **Vertex Cover** $\leq_P$ **Independent Set**.

The **Set Cover** Problem

Problem (**Set Cover**)

Input: *Given a set* U *of* n *elements, a collection* $S_1, S_2, \ldots S_m$ *of subsets of* U*, and an integer* k*.*

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Example

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$, $k = 2$ with

$$\begin{array}{ll} S_1 = \{3, 7\} & S_2 = \{3, 4, 5\} \\ S_3 = \{1\} & S_4 = \{2, 4\} \\ S_5 = \{5\} & S_6 = \{1, 2, 6, 7\} \end{array}$$

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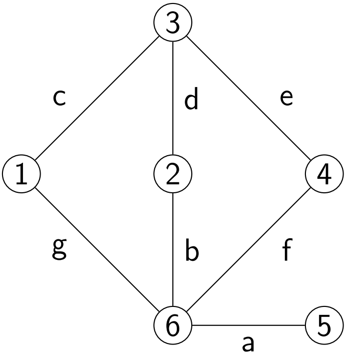
Observe that G has vertex cover of size k if and only if $U, \{S_v\}_{v \in V}$ has a set cover of size k. (Exercise: Prove this.)

Vertex Cover $\leq_P$ Set Cover: Example



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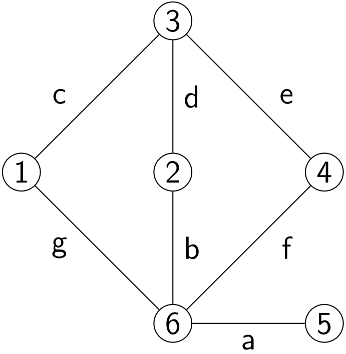

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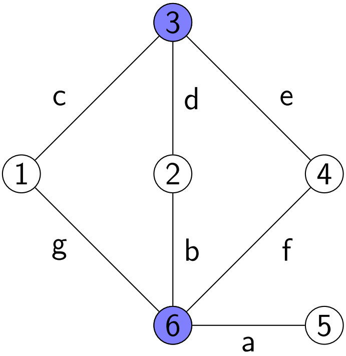

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$\{S_3, S_6\}$ is a set cover

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Proving Reductions

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3. Runs in **polynomial** time.

Example of incorrect reduction proof

Try proving **Matching** $\leq_P$ **Bipartite Matching** via following reduction:

1. Given graph $G = (V, E)$ obtain a bipartite graph $G' = (V', E')$ as follows.
   1. Let $V_1 = \{u_1 \mid u \in V\}$ and $V_2 = \{u_2 \mid u \in V\}$. We set $V' = V_1 \cup V_2$ (that is, we make two copies of V)

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2. Given G and integer k the reduction outputs G' and k.

“Proof”

Claim

Reduction is a poly-time algorithm. If G has a matching of size k then G' has a matching of size k.

Proof.

Exercise. □

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Incorrect! Why? Vertex $u \in V$ has two copies u_1 and u_2 in G'. A matching in G' may use both copies!

Subset sum and Partition?

Problem: Subset Sum

Instance: S - set of positive integers, t: - an integer number (target).
Question: Is there a subset $X \subseteq S$ such that $\sum_{x \in X} x = t$?

Problem: Partition

Instance: A set S of n numbers.
Question: Is there a subset $T \subseteq S$ s.t. $\sum_{t \in T} t = \sum_{s \in S \setminus T} s$?

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Assume that we can solve **Subset Sum** in polynomial time, then we can solve **Partition** in polynomial time. This statement is

(A) True.

(B) Mostly true.

(C) False.

(D) Mostly false.

II: Partition and subset sum?

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