Heuristics, Approximation Algorithms

Lecture 24

April 24, 2018

Most slides are courtesy Prof. Chekuri

Part I

Heuristics

Coping with Intractability

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3. Exploit properties of instances that arise in practice which may be much easier. Give up on hard instances, which is OK.
4. Settle for sub-optimal (aka approximate) solutions, especially for optimization problems

NP and EXP

EXP: all problems that have an exponential time algorithm.

Proposition

NP $\subseteq$ **EXP**.

Proof.

Let $X \in$ **NP** with certifier C. To prove $X \in EXP$, here is an algorithm for X. Given input s,

1. For every t, with $|t| \le p(|s|)$ run $C(s,t)$; answer "yes" if any one of these calls returns "yes", otherwise say "no". □

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Every problem in **NP** has a brute-force "try all possibilities" algorithm that runs in exponential time.

Examples

1. **SAT**: try all possible truth assignment to variables.
2. **Independent set**: try all possible subsets of vertices.
3. **Vertex cover**: try all possible subsets of vertices.

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Backtrack Search Algorithm for SAT

Input: CNF Formula φ on n variables $x_1, \ldots, x_n$ and m clauses
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6. Return "no"

Certain part of the search space is pruned.

Example

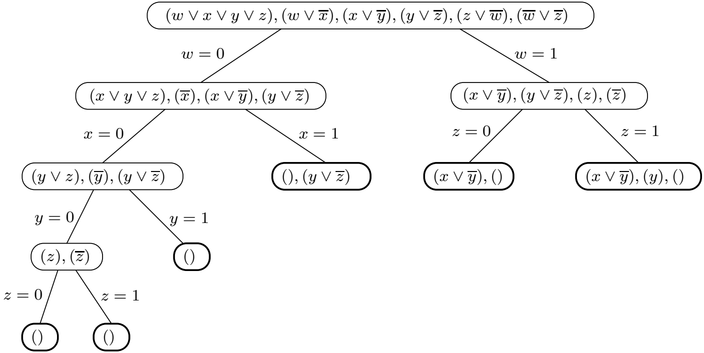

Figure: Backtrack search. Formula is not satisfiable.

Figure taken from Dasgupta etal book.

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2. Run obvious test and in addition if all clauses are of size **2** then run 2-SAT polynomial time algorithm
3. ...

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5. Output best solution found.

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How do we compute a lower bound?
One possibility: solve an LP relaxation.

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3. If there is a solution $s' \in N(s)$ that is better than s, move to s' and continue search with s'
4. Else, stop search and output s.

Local Search

Main ingredients in local search:

1. Initial solution.
2. Definition of neighborhood of a solution.
3. Efficient algorithm to find a good solution in the neighborhood.

Example: TSP

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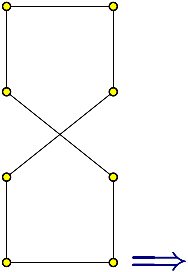
3. For a solution s at most $O(n^2)$ neighbors and one can try all of them to find an improvement.

TSP: 2-change example



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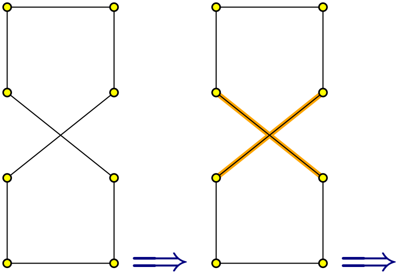

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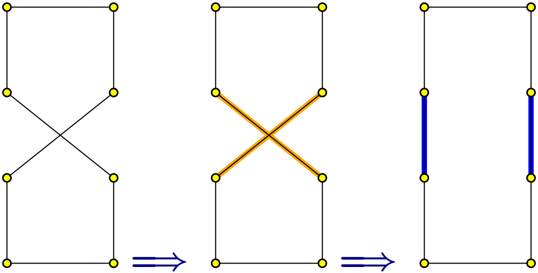

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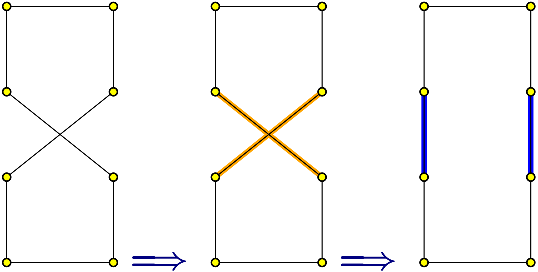

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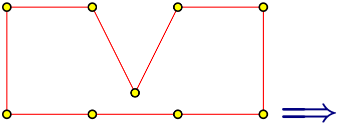

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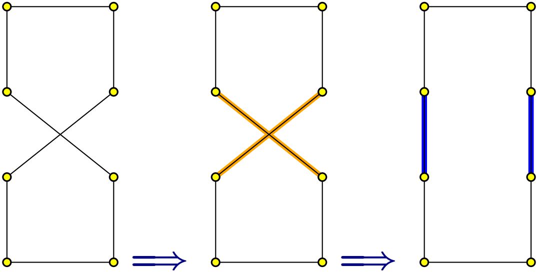

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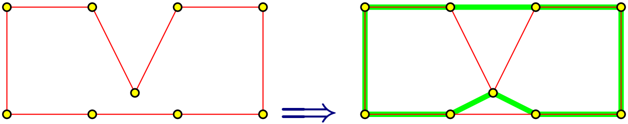
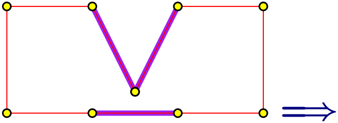


TSP: 3-change example

3-change local search: swap $\mathbf{3}$ edges out.



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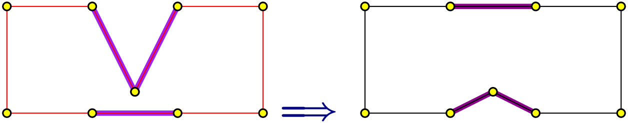

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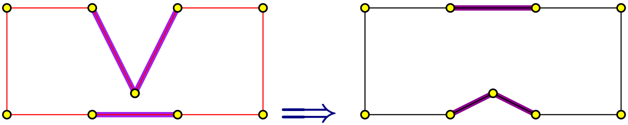

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Can define k-change heuristic where k edges are swapped out. Increases neighborhood size and makes each local improvement step less efficient.

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3. Tabu search. Store already visited solutions and do not visit them again (they are "taboo").

Heuristics

Several other heuristics used in practice.

1. Heuristics for solving integer linear programs such as cutting planes, branch-and-cut etc are quite effective. They exploit the geometry of the problem.
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Heuristics design is somewhat ad hoc and depends heavily on the problem and the instances that are of interest.

Part II

Approximation Algorithms

Approximation algorithms

Consider the following *optimization* problems:

1. **Max Knapsack**: Given knapsack of capacity W, n items each with a value and weight, pack the knapsack with the most profitable subset of items whose weight does not exceed the knapsack capacity.
2. **Min Vertex Cover:** given a graph $G = (V, E)$ find the minimum cardinality vertex cover.
3. **Min Set Cover:** given Set Cover instance, find the smallest number of sets that cover all elements in the universe.
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Solving one in polynomial time implies solving all the others.

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Informal definition: An approximation algorithm for an optimization problem is an efficient (polynomial-time) algorithm that *guarantees* for every instance a solution of some given quality when compared to an optimal solution.

Some known approximation results

1. **Knapsack**: For every fixed $\epsilon > 0$ there is a polynomial time algorithm that guarantees a solution of quality $(1 - \epsilon)$ times the best solution for the given instance. Hence can get a 0.99-approximation efficiently.

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5. **Min TSP**: No polynomial factor relative approximation possible.

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2. Approximation is a useful lens to examine **NP-Complete** problems more closely.
3. Approximation also useful for problems that we can solve efficiently:
    1. We may have other constraints such a space (streaming problems) or time (need linear time or less for very large problems)
    2. Data may be uncertain (online and stochastic problems).

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- Let $OPT(I)$ and $\mathcal{A}(I)$ denote the values of an optimum solution and the solution output by $\mathcal{A}$ on instances I. Then
  - If X is a minimization problem: $\mathcal{A}(I)/OPT(I) \leq \alpha$
  - If X is a maximization problem: $OPT(I)/\mathcal{A}(I) \leq \alpha$

Formal definition of approximation algorithm

An algorithm $\mathcal{A}$ for an optimization problem X is an α-approximation algorithm if the following conditions hold:

- for each instance I of X the algorithm $\mathcal{A}$ correctly outputs a valid solution to I
- $\mathcal{A}$ is a polynomial-time algorithm
- Let $OPT(I)$ and $\mathcal{A}(I)$ denote the values of an optimum solution and the solution output by $\mathcal{A}$ on instances I. Then
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To be formal we need to say $\alpha(n)$ where $n = |I|$ since in some cases the *approximation ratio* depends on the size of the instance.

Formal definition of approximation algorithm

Unfortunately notation is not consistently used. Some times people use the following convention:

- If X is a minimization problem then $\mathcal{A}(I)/OPT(I) \leq \alpha$ and here $\alpha \geq 1$.
- If X is a maximization problem then $\mathcal{A}(I)/OPT(I) \geq \alpha$ and here $\alpha \leq 1$.

Usually clear from the context.

Relative vs Additive

We defined approximation ratio in a relative sense. Some times it makes sense to ask for an additive approximation. For instance in continuous optimization such as linear/convex optimization we talk about ϵ-error where we want a solution I such that $|\mathcal{A}(I) - OPT(I)| \leq \epsilon$.

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For most NP-Hard optimization problems it is not hard to show that one cannot obtain a good additive approximation in polynomial time unless $P = NP$ and hence relative approximation is a more robust and useful notion.

Part III

Approximation for Vertex Cover

Vertex Cover

Given a graph $G = (V, E)$, a set of vertices S is:

1. A **vertex cover** if every $e \in E$ has at least one endpoint in S.

Problem (**Vertex Cover**)

Input: *A graph* G

Goal: *Find a vertex cover of minimum size in* G

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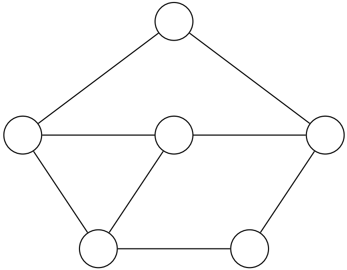

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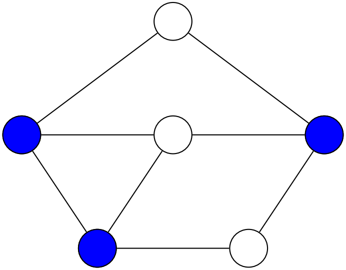

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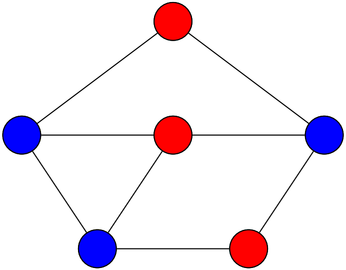

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Initialize S to be ∅
While there are edges in G do
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Theorem

There is an infinite family of graphs where the solution S output by Greedy is $\Omega(\ln n)OPT$.

Matching Heuristic

Relation between matching and vertex cover

Lemma

Let $M \subset E$ *be a matching in graph* $G = (V, E)$*, then* $OPT \geq |M|$ *where* OPT *is the size of minimum vertex cover.*

Matching Heuristic

Relation between matching and vertex cover

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Lemma

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Analysis: $|S| = 2|M| \leq 2OPT$. Algorithm is a 2-approximation.

Vertex Cover: LP Relaxation based approach

Write (weighted) vertex cover problem as an integer linear program

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\begin{array}{lll}
\text{Minimize} & \sum_{v \in V} w_v x_v & \\
\text{subject to} & x_u + x_v \geq 1 & \text{for each } uv \in E \\
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Lemma

$OPT \geq \sum_v w_v x_v^*$.

Vertex Cover: Rounding fractional solution

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Rounding: $S = \{v \mid x^*_v \geq 1/2\}$. Output S.

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Set Cover and Vertex Cover

Theorem

Greedy gives $(\ln n + 1)$*-approximation for Set Cover where* n *is number of elements.*

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Unless $P = NP$ *no* $(\ln n + \epsilon)$*-approximation for Set Cover for* $\epsilon < 1$.

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Unless $P = NP$ no 1.36-approximation for Vertex Cover.

Conjecture: Unless $P = NP$ no $(2 - \epsilon)$-approximation for Vertex Cover for any fixed $\epsilon > 0$.

Independent Set and Vertex Cover

Proposition

Let $G = (V, E)$ *be a graph.* S *is an independent set if and only if* $V \setminus S$ *is a vertex cover.*

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If S^* is a minimum sized vertex cover then $V - S^*$ is a max independent set.

Independent Set and Vertex Cover

IndependentSetHeuristic$(G = (V, E))$:
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- Let k be minimum vertex cover size.
- Suppose $k = n/2$ where $n = |V|$
- Then V is a 2-approximation
- But then algorithm will output an **empty** independent set even though there is an independent set of size $n/2$.

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Example?

Theorem

Unless $P = NP$ no $n^{1-\delta}$-approximation for Independent Set for any fixed $\delta > 0$.