More Approximation Algorithms

Lecture 25

April 26, 2018

Most slides are courtesy Prof. Chekuri

Formal definition of approximation algorithm

An algorithm $\mathcal{A}$ for an optimization problem X is an α-approximation algorithm if the following conditions hold:

- for each instance I of X the algorithm $\mathcal{A}$ correctly outputs a valid solution to I
- $\mathcal{A}$ is a polynomial-time algorithm
- Letting $OPT(I)$ and $\mathcal{A}(I)$ denote the values of an optimum solution and the solution output by $\mathcal{A}$ on instances I, $OPT(I)/\mathcal{A}(I) \leq \alpha$ *and* $\mathcal{A}(I)/OPT(I) \leq \alpha$. Alternatively:
  - If X is a minimization problem: $\mathcal{A}(I)/OPT(I) \leq \alpha$
  - If X is a maximization problem: $OPT(I)/\mathcal{A}(I) \leq \alpha$

Definition ensures that $\alpha \geq 1$

To be formal we need to say $\alpha(n)$ where $n = |I|$ since in some cases the *approximation ratio* depends on the size of the instance.

Part I

Approximation for Load Balancing

Load Balancing

Given n jobs $J_1, J_2, \ldots, J_n$ with sizes $s_1, s_2, \ldots, s_n$ and m identical machines $M_1, \ldots, M_m$ assign jobs to machines to minimize maximum load (also called makespan).

Problem sometimes referred to as multiprocessor scheduling.
Example: $\mathbf{3}$ machines and $\mathbf{8}$ jobs with sizes $\mathbf{4, 3, 1, 2, 5, 6, 9, 7}$.

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Formally, an assignment is a mapping $f : \{1, 2, \ldots, n\} \to \{1, \ldots, m\}$.

- The load $\ell_f(j)$ of machine M_j under f is $\sum_{i:f(i)=j} s_i$
- Goal is to find f to minimize $\max_j \ell_f(j)$.

Greedy List Scheduling

List-Scheduling

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Let J_1, J_2, ..., J_n be an ordering of jobs
for i = 1 to n do
    Schedule job J_i on the currently least loaded machine
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Different list: $\mathbf{9, 7, 6, 5, 4, 3, 2, 1}$

Two lower bounds on OPT

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Lower bounds on OPT:

- average load: $OPT \geq \sum_{i=1}^{n} s_i/m$. Why?
- maximum job size: $OPT \geq \max_{i=1}^{n} s_i$. Why?

Analysis of Greedy List Scheduling

Theorem

Let L be makespan of Greedy List Scheduling on a given instance. Then $L \leq 2(1 - 1/m)OPT$ where OPT is the optimum makespan for that instance.

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- Let M_h be the machine which achieves the load L for Greedy List Scheduling.
- Let J_i be the job that was last scheduled on M_h.
- Why was J_i scheduled on M_h? It means that M_h was the least loaded machine when J_i was considered. Implies all machines had load at least $L - s_i$ at that time.

Analysis continued

Lemma

$L - s_i \leq (\sum_{\ell=1}^{i-1} s_\ell)/m$.

Proof.

Since all machines had load at least $L - s_i$ it means that $m(L - s_i) \leq \sum_{\ell=1}^{i-1} s_\ell$ and hence

$$L - s_i \leq (\sum_{\ell=1}^{i-1} s_\ell)/m.$$

□

Analysis continued

But then

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\begin{aligned}
L &\leq (\sum_{\ell=1}^{i-1} s_\ell)/m + s_i \\
&\leq (\sum_{\ell=1}^{n} s_\ell)/m + (1 - \frac{1}{m})s_i \\
&\leq
\end{aligned}
$$

Analysis continued

But then

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\begin{aligned}
L &\leq (\sum_{\ell=1}^{i-1} s_\ell)/m + s_i \\
&\leq (\sum_{\ell=1}^{n} s_\ell)/m + (1 - \frac{1}{m})s_i \\
&\leq OPT + (1 - \frac{1}{m})OPT \\
&\leq (2 - \frac{2}{m})OPT \\
&= 2(1 - \frac{1}{m})OPT.
\end{aligned}
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Question: Is the analysis of the algorithm tight? That is, are there instances where L is $2(1 - 1/m)OPT$?

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- $OPT = m$. Why?
- If the list has large job at end the Greedy will give makespan of $m + m - 1 = 2m - 1$.

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Theorem

Greedy List Scheduling with jobs sorted from largest to smallest gives a $\mathbf{4/3}$*-approximation and this is essentially tight.*

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Need another lower bound

Another useful lower bound

Lemma

Suppose jobs are sorted, that is $s_1 \geq s_2 \geq \ldots \geq s_n$ *and* $n > m$ *then* $OPT \geq s_m + s_{m+1} \geq 2s_{m+1}$.

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Proof.

Consider OPTimal schedule of the first $m+1$ jobs $J_1, \ldots, J_{m+1}$.
By pigeon hole principle two of these jobs on same machine.

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Consider OPTimal schedule of the first $m + 1$ jobs $J_1, \ldots, J_{m+1}$. By pigeon hole principle two of these jobs on same machine.

$OPT \geq$ Load on that machine $\geq$ the sum of the smallest two job sizes in the first $m + 1$ jobs

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$OPT \geq$ Load on that machine $\geq$ the sum of the smallest two job sizes in the first $m+1$ jobs $= s_m + s_{m+1}$. $\square$

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- Case 1: If s_i is only job on M_j then $L = s_i \leq OPT$.
- Case 2: At least one more job on M_j before s_i.
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- Case 2: At least one more job on M_j before s_i.
  - We have seen that $L - s_i \leq OPT$.
  - **Claim:** $s_i \leq OPT/2$
  - Together, we have $L \leq OPT + s_i \leq 3OPT/2$.

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Since M_j had a job before s_i we have $i > m$. Why?

Hence $s_i \leq s_{m+1}$ becase jobs were sorted by decreasing size. Since $2s_{m+1} \leq OPT$, we have $s_i \leq s_{m+1} \leq OPT/2$.

Part II

Approximation for Set Cover

Set Cover

Input: Universe $\mathcal{U}$ of n elements and m subsets $S_1, S_2, \ldots, S_m$ such that $\cup_i S_i = \mathcal{U}$.

Goal: Pick fewest number of subsets to cover all of $\mathcal{U}$ (equivalently, whose union is $\mathcal{U}$).

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Greedy$(\mathcal{U}, S_1, S_2, \ldots, S_m)$

```
Uncovered = U
While Uncovered ≠ ∅ do
    Pick set S_j that covers max number of uncovered elements
    Add S_j to solution
    Uncovered = Uncovered − S_j
endWhile
Output chosen sets
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Analysis of Greedy

- Let k^* be minimum number of sets to cover $\mathcal{U}$. Let k be number of sets chosen by Greedy.
- Let α_i be # new elements covered in iteration i.
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There exists one of those sets that covers at least $\mathcal{U}_i/k^*$ elements.
Greedy picks the best set and hence covers at least that many elements. Note $\mathcal{U}_i = \beta_{i-1}$. □

Analysis of Greedy contd

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Hence by induction,

$$\beta_i \leq \beta_0(1 - 1/k^*)^i = n(1 - 1/k^*)^i.$$

Thus, after $k = k^* \ln n$ iterations number number of uncovered elements is at most

$$n(1 - 1/k^*)^{k^* \ln n} \leq ne^{-\ln n} \leq 1.$$

Analysis of Greedy contd

Lemma

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Thus algorithm terminates in at most $k^* \ln n + 1$ iterations. Total number of sets chosen is number of iterations.

Analysis contd

Theorem

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- Can show a tighter bound of $(\ln d + 1)$ where d is maximum set size.

Theorem

Unless $P = NP$ no $(\ln n + \epsilon)$-approximation for Set Cover.

A bad example for Greedy

$n = 2(1 + 2 + 2^2 + \cdots + 2^p) = 2(2^{p+1} - 1)$, $m = 2 + (p + 1)$, $OPT = 2$, Greedy picks $(p + 1)$ and hence ratio is $\Omega(\ln n)$.

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Greedy is a simple algorithm. In several scenarios the set system is *implicit* and exponentially large in n.

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Example. Covering all the edges of a graph using minimum number of disjoint trees.

Max k-Cover

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Goal: Pick k subsets to *maximize* number of covered elements.

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    endWhile
    Output chosen k sets
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Analysis

Similar to previous analysis.

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- Let $\beta_i = OPT - \gamma_i$. Define $\beta_0 = OPT$.

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Lemma

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Proof: Exercise.

Analysis contd

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Hence by induction,

$$\beta_i \leq \beta_0(1 - 1/k)^i = OPT(1 - 1/k)^i.$$

Thus, after k iterations,

$$\beta_k \leq OPT(1 - 1/k)^k \leq OPT/e.$$

Analysis contd

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Thus, after k iterations,

$$\beta_k \leq OPT(1 - 1/k)^k \leq OPT/e.$$

Thus $\gamma_k = OPT - \beta_k \geq (1 - 1/e)OPT$.

Analysis contd

Theorem

Greedy gives a $(1 - 1/e)$-approximation for Max k-Coverage.

Above theorem generalizes to submodular function maximization and has *many* applications.

Theorem (Feige 1998)

Unless $P = NP$ there is no $(1 - 1/e - \epsilon)$-approximation for Max k-Coverage for any fixed $\epsilon > 0$.