Probabilistic Inequalities and Examples

Lecture 3

January 22, 2019

Outline

Probabilistic Inequalities

Markov's Inequality

Chebyshev's Inequality

Bernstein-Chernoff-Hoeffding bounds

Some examples

Part I

Inequalities

Massive randomness.. Is not that random.

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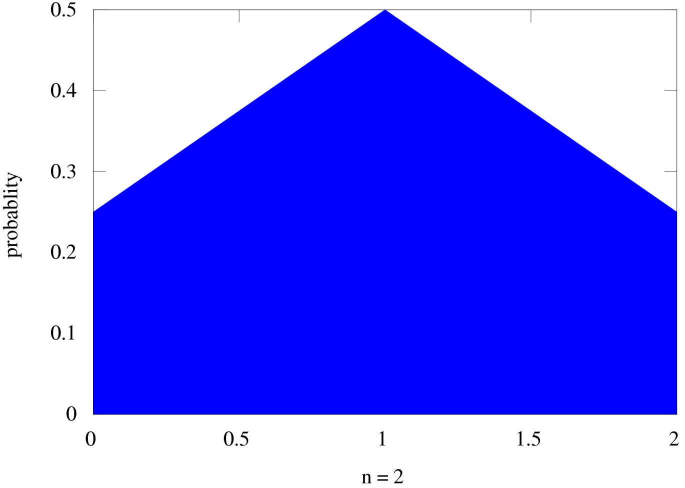

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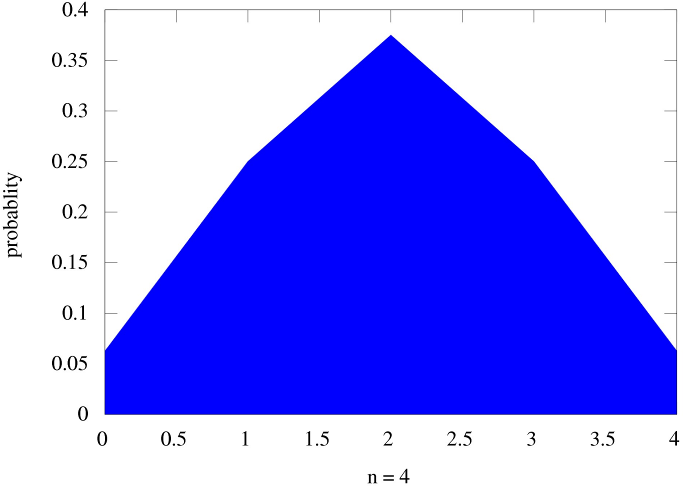

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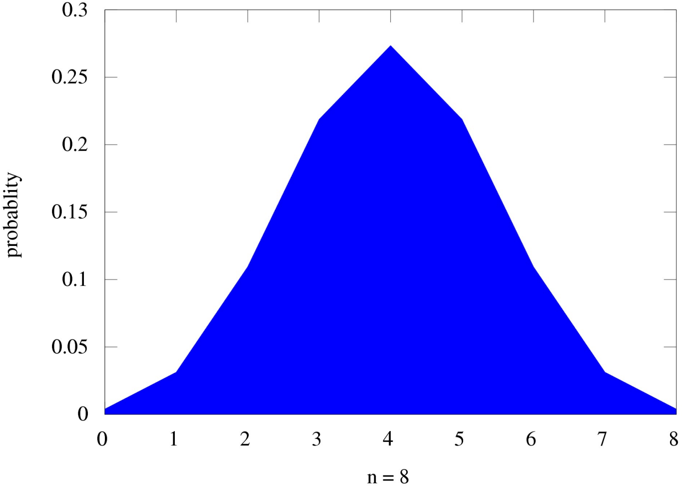

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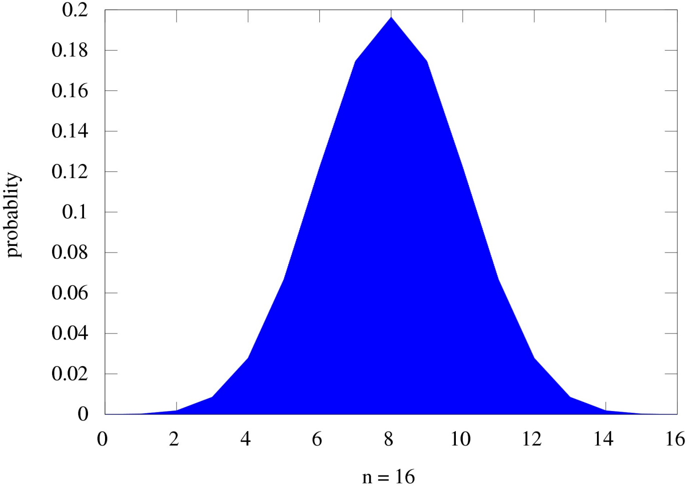

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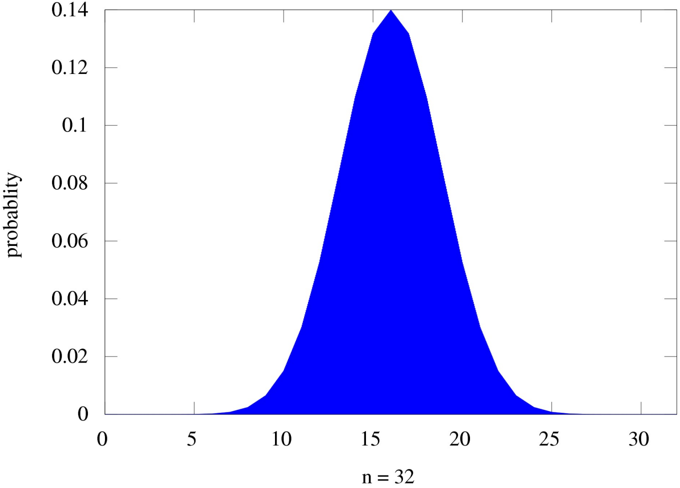

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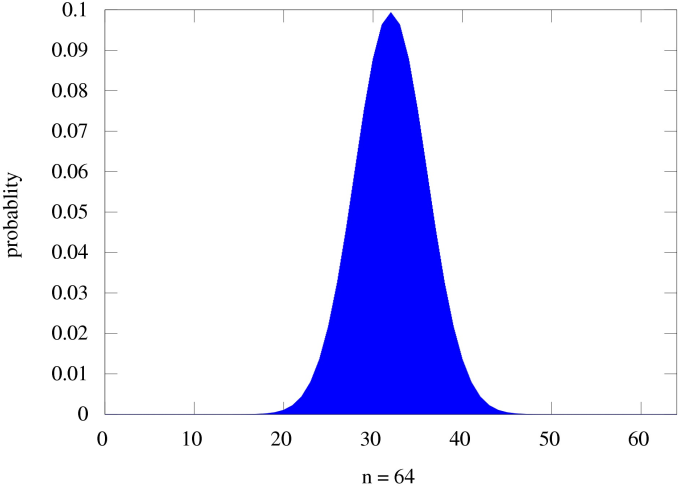

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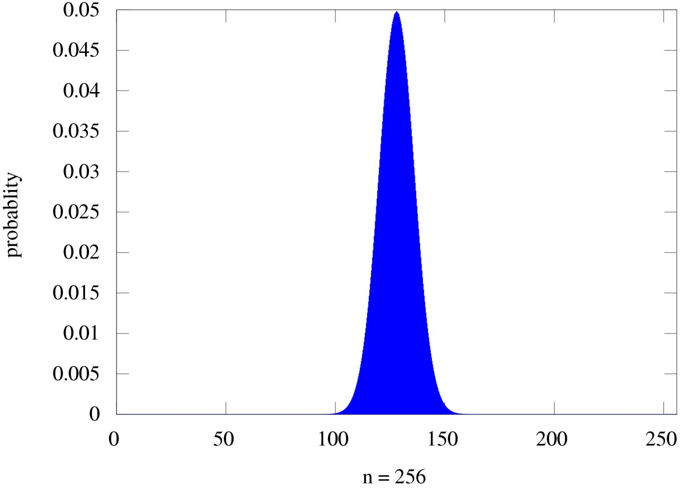

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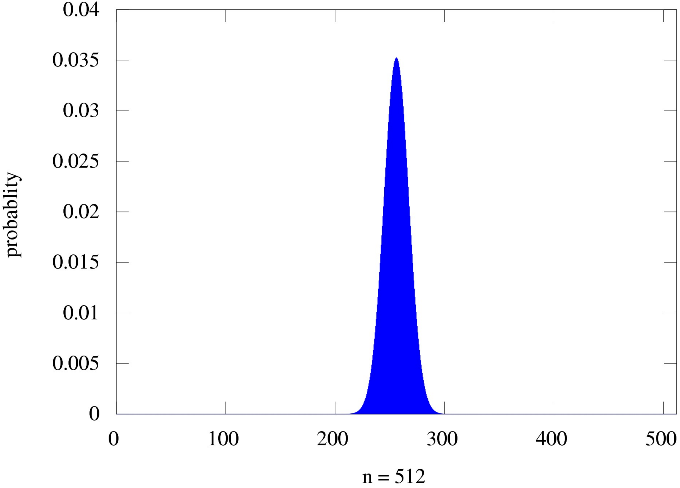

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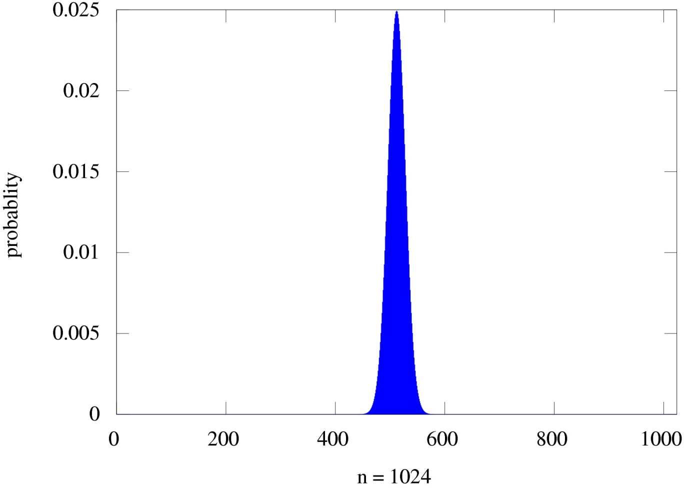

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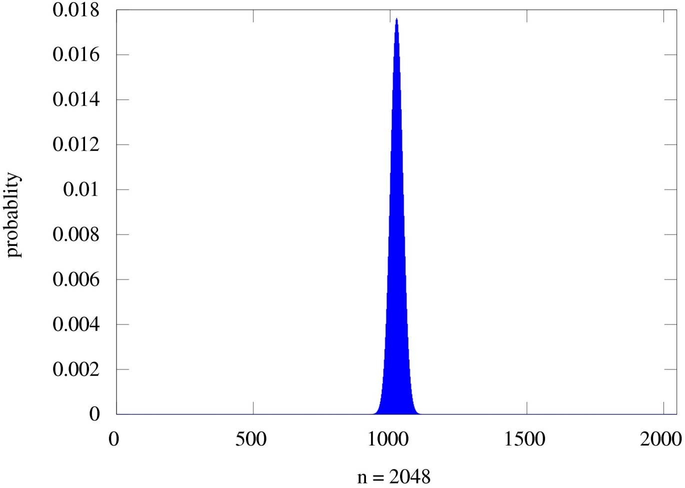

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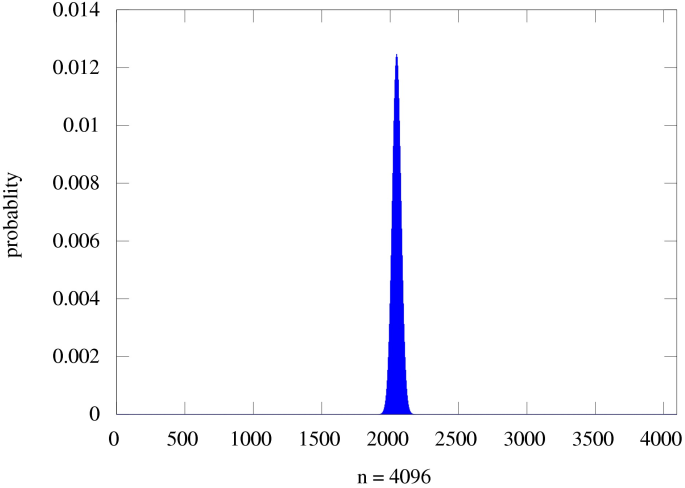

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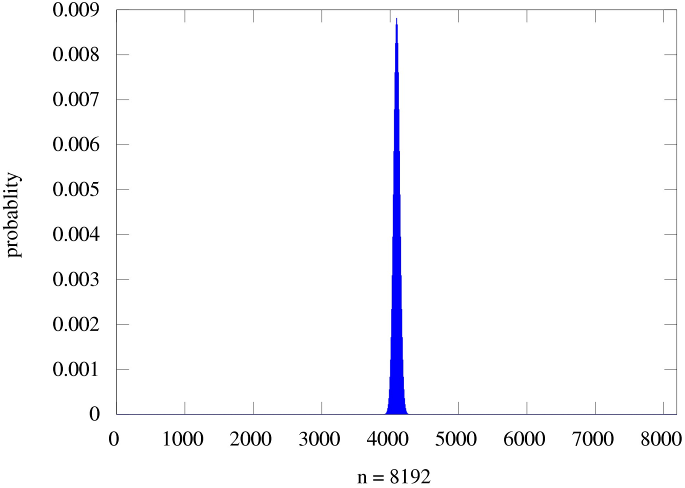

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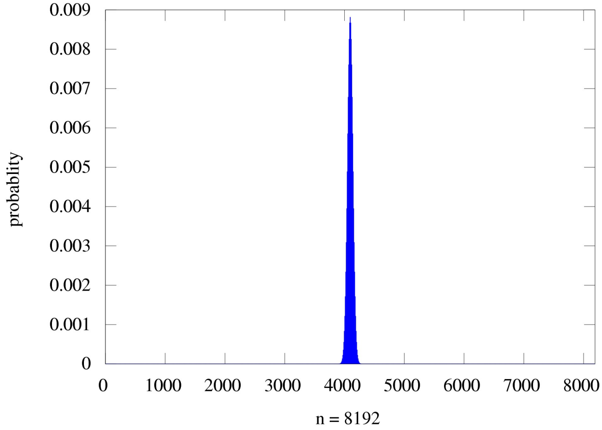
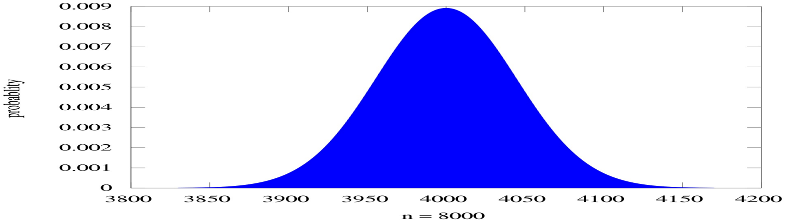


This is known as **concentration of mass**.
This is a very special case of the **law of large numbers**.

Informal statement of law of large numbers

For n large enough, the middle portion of the binomial distribution looks like (converges to) the normal/Gaussian distribution.



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Intuitive conclusion

Randomized algorithm are unpredictable in the tactical level, but very predictable in the strategic level.

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Use of well known inequalities in analysis.

Randomized **QuickSort**: A possible analysis

Analysis

- Random variable $Q = \#comparisons$ made by randomized **QuickSort** on an array of n elements.

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- $\mathbf{E}[Q] \leq 10n\log n + (n^2 - 10n\log n)c$.

Question:

How to find c, or in other words bound $\Pr[Q \geq 10n\log n]$?

Markov's Inequality

Markov's inequality

Let X be a **non-negative** random variable over a probability space $(\Omega, \Pr)$. For any $a > 0$, $\Pr[X \geq a] \leq \frac{E[X]}{a}$. Equivalently, for any $t > 0$, $\Pr[X \geq tE[X]] \leq 1/t$.

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Proof:

$$
\begin{aligned}
\mathbf{E}[X] &= \sum_{\omega \in \Omega} X(\omega) \Pr[\omega] \\
&= \sum_{\omega,\ 0 \leq X(\omega) < a} X(\omega) \Pr[\omega] + \sum_{\omega,\ X(\omega) \geq a} X(\omega) \Pr[\omega] \\
&\geq \sum_{\omega \in \Omega,\ X(\omega) \geq a} X(\omega) \Pr[\omega] \\
&\geq a \sum_{\omega \in \Omega,\ X(\omega) \geq a} \Pr[\omega] \\
&= a \Pr[X \geq a]
\end{aligned}
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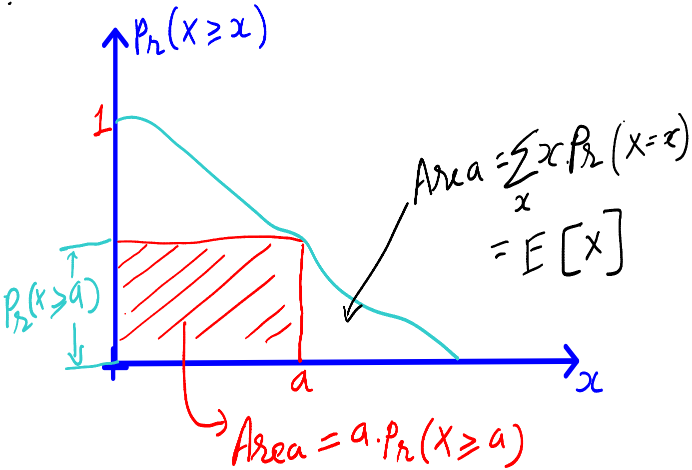
Proof:

$$\begin{aligned} \mathbf{E}[X] &= \int_0^\infty z f_X(z) dz \\ &\geq \int_a^\infty z f_X(z) dz \\ &\geq a \int_a^\infty f_X(z) dz \\ &= a \,\mathbf{Pr}[X \geq a] \end{aligned}$$

Markov's Inequality: Proof by Picture



Chebyshev's Inequality: Variance

Variance

Given a random variable X over probability space $(\Omega, \Pr)$, variance of X is the measure of how much does it deviate from its mean value. Formally, $Var(X) = \mathbf{E}\left[(X - \mathbf{E}[X])^2\right] = \mathbf{E}\left[X^2\right] - \mathbf{E}[X]^2$

Derivation

Define $Y = (X - \mathbf{E}[X])^2 = X^2 - 2X\,\mathbf{E}[X] + \mathbf{E}[X]^2$.

$$\begin{aligned} Var(X) &= \mathbf{E}[Y] \\ &= \mathbf{E}\left[X^2\right] - 2\,\mathbf{E}[X]\,\mathbf{E}[X] + \mathbf{E}[X]^2 \\ &= \mathbf{E}\left[X^2\right] - \mathbf{E}[X]^2 \end{aligned}$$

Chebyshev's Inequality: Variance

Independence

Random variables X and Y are called mutually independent if

$$\forall x, y \in \mathbb{R},\ \mathbf{Pr}[X = x \wedge Y = y] = \mathbf{Pr}[X = x]\,\mathbf{Pr}[Y = y]$$

Lemma

If X and Y are independent random variables then $Var(X + Y) = Var(X) + Var(Y)$.

Chebyshev's Inequality: Variance

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Lemma

If X and Y are mutually independent, then $\mathbf{E}[XY] = \mathbf{E}[X]\,\mathbf{E}[Y]$.

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If $Var X < \infty$, for any $a \geq 0$, $\Pr[|X - \mathbf{E}[X]| \geq a] \leq \frac{Var(X)}{a^2}$

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Proof.

$Y = (X - \mathbf{E}[X])^2$ is a non-negative random variable. Apply Markov's Inequality to Y for a^2.

$$\Pr\left[Y \geq a^2\right] \leq {}^{\mathbf{E}[Y]}/_{a^2} \Leftrightarrow \Pr\left[(X - \mathbf{E}[X])^2 \geq a^2\right] \leq {}^{\mathit{Var}(X)}/_{a^2}$$
$$\Leftrightarrow \Pr[|X - \mathbf{E}[X]| \geq a] \leq {}^{\mathit{Var}(X)}/_{a^2}$$

$\square$

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$$\begin{aligned}\Pr\left[Y \geq a^2\right] \leq \mathbf{E}[Y]/a^2 &\Leftrightarrow \Pr\left[(X - \mathbf{E}[X])^2 \geq a^2\right] \leq Var(X)/a^2 \\ &\Leftrightarrow \Pr[|X - \mathbf{E}[X]| \geq a] \leq Var(X)/a^2\end{aligned}$$

$\square$

$\Pr[X \leq \mathbf{E}[X] - a] \leq Var(X)/a^2$ AND $\Pr[X \geq \mathbf{E}[X] + a] \leq Var(X)/a^2$

Chebyshev's Inequality

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Given $a \geq 0$, $\mathbf{Pr}[|X - \mathbf{E}[X]| \geq a] \leq \frac{Var(X)}{a^2}$ equivalently for any $t > 0$, $\mathbf{Pr}[|X - \mathbf{E}[X]| \geq t\sigma_X] \leq \frac{1}{t^2}$ where $\sigma_X = \sqrt{Var(X)}$ is the standard deviation of X.

Example: Random walk on the line

- Start at origin 0. At each step move left one unit with probability $1/2$ and move right with probability $1/2$.
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By Chebyshev: $\mathbf{Pr}\left[|Y_n| \geq t\sqrt{n}\right] \leq 1/t^2$

Chernoff Bound: Motivation

In many applications we are interested in X which is sum of *independent* bounded random variables.

$X = \sum_{i=1}^{k} X_i$ where $X_i \in [0, 1]$ or $[-1, 1]$ (normalizing)

Chebyshev not strong enough. For random walk on line one can prove

$$\Pr\left[|Y_n| \geq t\sqrt{n}\right] \leq 2exp(-t^2/2)$$

Chernoff Bound: Non-negative case

Lemma

Let $X_1, \ldots, X_k$ *be* k *independent binary random variables such that, for each* $i \in [1, k]$, $\mathbf{E}[X_i] = \Pr[X_i = 1] = p_i$. *Let* $X = \sum_{i=1}^{k} X_i$. *Then* $\mathbf{E}[X] = \sum_i p_i$.

- *Upper tail bound: For any* $\mu \geq \mathbf{E}[X]$ *and any* $\delta > 0$,

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

- *Lower tail bound: For any* $0 < \mu < \mathbf{E}[X]$ *and any* $0 < \delta < 1$,

$$\Pr[X \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\mu}$$

Chernoff Bound: Non-negative case, simplifying

When $0 < \delta < 1$ an important regime of interest we can simplify.

Lemma

Let $X_1, \ldots, X_k$ be k independent random variables such that, for each $i \in [1, k]$, X_i equals 1 with probability p_i, and 0 with probability $(1 - p_i)$. Let $X = \sum_{i=1}^{k} X_i$ and $\mu = \mathbf{E}[X] = \sum_i p_i$. For any $0 < \delta < 1$, it holds that:

$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{\frac{-\delta^2\mu}{3}}$$

$$\Pr[X \geq (1 + \delta)\mu] \leq e^{\frac{-\delta^2\mu}{3}} \text{ and } \Pr[X \leq (1 - \delta)\mu] \leq e^{\frac{-\delta^2\mu}{2}}$$

Chernoff Bound: general

Lemma

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$$\Pr[|X - \mathbf{E}[X]\,| \geq a] \leq 2exp(\frac{-a^2}{2n}).$$

When variables are not positive the bound depends on n while in the non-negative case there is no dependence on n (dimension-free)

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Applying to random walk:

$$\Pr\left[|Y_n| \geq t\sqrt{n}\right] \leq 2exp(-t^2/2).$$

Chernoff Bounds

Many variations and generalization that are useful in specific situations. See pointers on course webpage.

Part II

Ball and Bins

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- m balls and n bins
- Each ball thrown independently and uniformly in a bin
- Want to understand properties of bin loads
- Fundamental problem with many applications

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- Z_{ij} indicator for ball i falling into bin j
- $X_j = \sum_{i=1}^{m} Z_{ij}$ is number of balls in bin j
- $\sum_{j=1}^{n} Z_{ij} = 1$ deterministically
- $\mathbf{E}[Z_{ij}] = 1/n$ for all i, j, and hence $\mathbf{E}[X_j] = m/n$ for each bin j

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Question: Suppose we throw n balls into n bins. What is the expectation of the *maximum* load?

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Theorem

Let $Y = \max_{j=1}^{n} X_j$ *be the maximum load. Then* $\Pr[Y > 10 \ln n / \ln \ln n] < 1/n^2$ *(high probability) and hence* $\mathbf{E}[Y] = O(\ln n / \ln \ln n)$.

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Proof technique: combine Chernoff bound and union bound which is powerful and general template

Maximum Load

Focus on bin 1 without loss of generality since bins are symmetric. Simplifying notation $X = \sum_i Z_i$ where X is load of bin 1 and Z_i is indicator of ball i falling in bin.

- Want to know $\Pr[X \geq 10 \ln n / \ln\ln n]$
- $\mu = \mathrm{E}[X] = 1$
- $(1+\delta) = 10 \ln n / \ln\ln n$. We are in large δ setting
- Apply the Chernoff upper tail bound:

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\mu}$$

- Calculate/simplify and see that $\Pr[X \geq 10 \ln n / \ln\ln n] \leq 1/n^3$

Maximum load

- For each bin j, $\Pr[X_j \geq 10 \ln n / \ln \ln n] \leq 1/n^3$
- Let A_j be event that $X_j \geq 10 \ln n / \ln \ln n$
- By union bound

$$\Pr[\cup_j A_j] \leq \sum_j \Pr[A_j] \leq n \cdot 1/n^3 \leq 1/n^2.$$

- Hence, with probability at least $(1 - 1/n^2)$ **no bin** has load more than $10 \ln n / \ln \ln n$.

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- Hence, with probability at least $(1 - 1/n^2)$ **no bin** has load more than $10 \ln n / \ln \ln n$.
- Let $Y = \max_j X_j$. $Y \leq n$. Hence

$$\mathbf{E}[Y] \leq (1 - 1/n^2)(10 \ln n / \ln \ln n) + (1/n^2) n.$$

From a ball's perspective

Consider a ball i. How many other balls fall into the same bin as i?

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- Ball i is thrown first wlog. And lands in some bin j.
- Then the other $n - 1$ balls are thrown.
- Now bin j is fixed. Hence expected load on bin j is $(1 - 1/n)$.
- What is variance? What is a high probability bound?

Part III

Approximate Median

Approximate median

- **Input:** n distinct numbers $a_1, a_2, \ldots, a_n$ and $0 < \epsilon < 1/2$
- **Output:** A number x from input such that $(1-\epsilon)n/2 \leq rank(x) \leq (1+\epsilon)n/2$

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Algorithm:

- Sample with replacement k numbers from $a_1, a_2, \ldots, a_n$
- Output median of the sampled numbers

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- Output median of the sampled numbers

Theorem

For any $0 < \epsilon < 1/2$ and $0 < \delta < 1$, if $k = O(\frac{1}{\epsilon^2} \log(1/\delta))$, the algorithm outputs an ϵ-approximate median with probability at least $(1 - \delta)$.

Approximate median

- Let S be random sample chosen by algorithm
- Imagine sorting the numbers
- Split numbers into L (left), M (middle), and R (right)
- $M = \{y \mid (1-\epsilon)n/2 \leq rank(y) \leq (1+\epsilon)n/2\}$
- Algorithm makes a mistake only if $|S \cap L| \geq k/2$ or $|S \cap R| \geq k/2$. Otherwise it will output a number from M.

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- Algorithm makes a mistake only if $|S \cap L| \geq k/2$ or $|S \cap R| \geq k/2$. Otherwise it will output a number from M.

Analysis:

- Let $Y = |S \cap L|$? What is $\mathbf{E}[Y]$?
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- By union bound at most δ probability that $|S \cap L| \geq k/2$ or $|S \cap R| \geq k/2$.
- Hence with $(1-\delta)$ probability median of S is an ϵ-approximate median

Part IV

Randomized **QuickSort** (Contd.)

Randomized **QuickSort**: Recall

Input: Array A of n numbers. **Output:** Numbers in sorted order.

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1. Pick a pivot element *uniformly at random* from A.
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Question: With what probability it takes $O(n \log n)$ time?

Randomized **QuickSort**: High Probability Analysis

Informal Statement

Random variable $Q(A) = \#$ comparisons done by the algorithm.

We will show that $\Pr[Q(A) \leq 32 n \ln n] \geq 1 - 1/n^3$.

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If $n = 100$ then this gives $\Pr[Q(A) \leq 32 n \ln n] \geq 0.99999$.

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  2. By union bound, any of the n elements participates in $> 32 \ln n$ levels with probability at most $1/n^3$.
  3. Therefore, all elements participate in $\leq 32 \ln n$ w.p. $(1 - 1/n^3)$.

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- If $\rho =$#lucky rounds in first k rounds, then $|S_k| \leq (3/4)^{\rho} n$.
- For $|S_k| = 1$, $\rho = 4 \ln n \geq \log_{4/3} n$ suffices.

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$$\begin{aligned}
\Pr[\rho \leq 4 \ln n] &= \Pr[\rho \leq k/8] \\
&= \Pr[\rho \leq (1-\delta)\mu] \\
(\textit{Chernoff}) \quad &\leq e^{\frac{-\delta^2 \mu}{2}} \\
&= e^{-\frac{9k}{64}} \\
&= e^{-4.5 \ln n} \leq \frac{1}{n^4}
\end{aligned}$$

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- n input elements. Probability that depth of recursion in **QuickSort** $> 32 \ln n$ is at most $\frac{1}{n^4} * n = \frac{1}{n^3}$.

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Theorem

With high probability (i.e., $1 - \frac{1}{n^3}$) the depth of the recursion of **QuickSort** *is* $\leq 32 \ln n$. *Due to* n *comparisons in each level, with high probability, the running time of* **QuickSort** *is* $O(n \ln n)$.

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