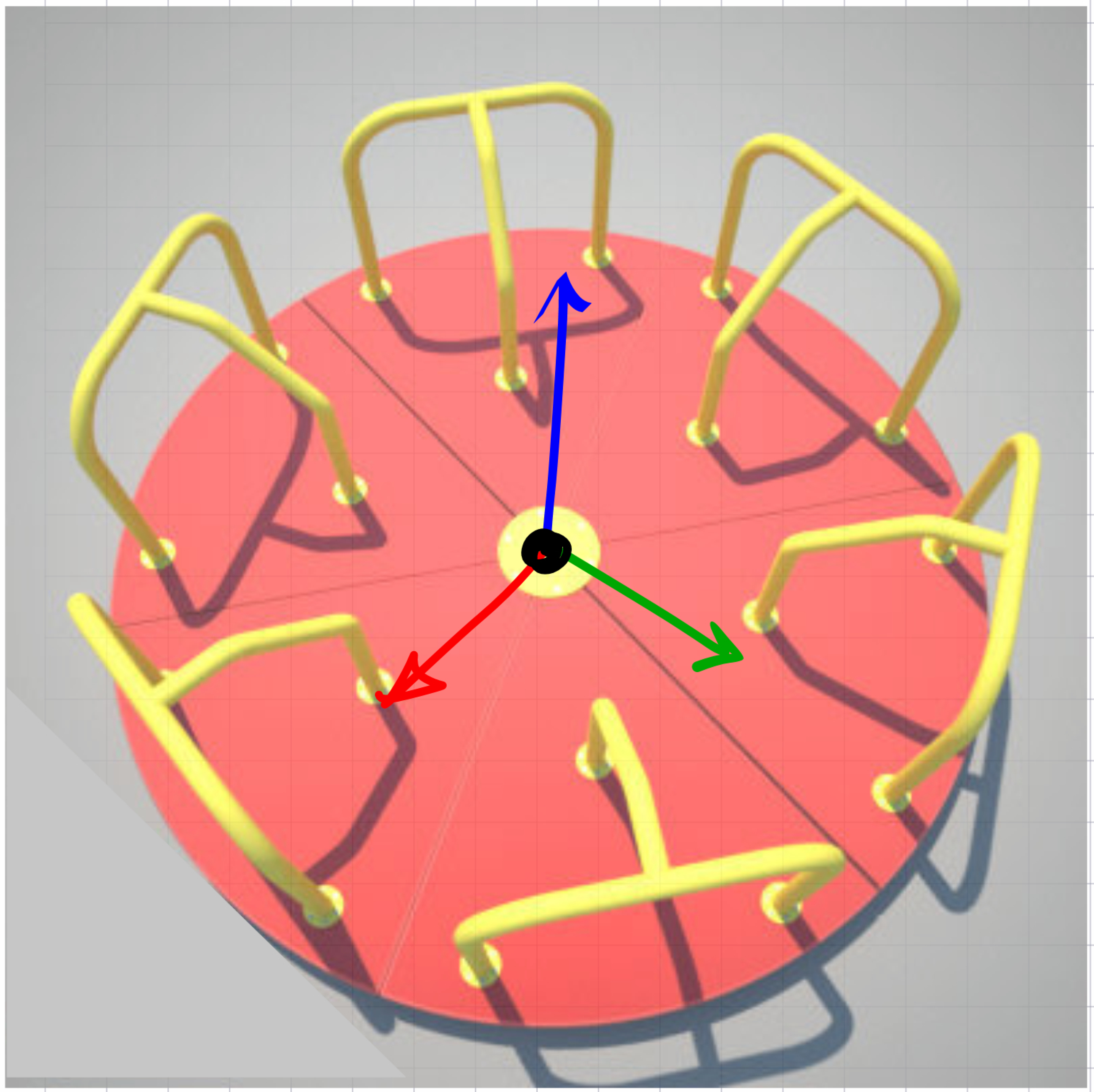
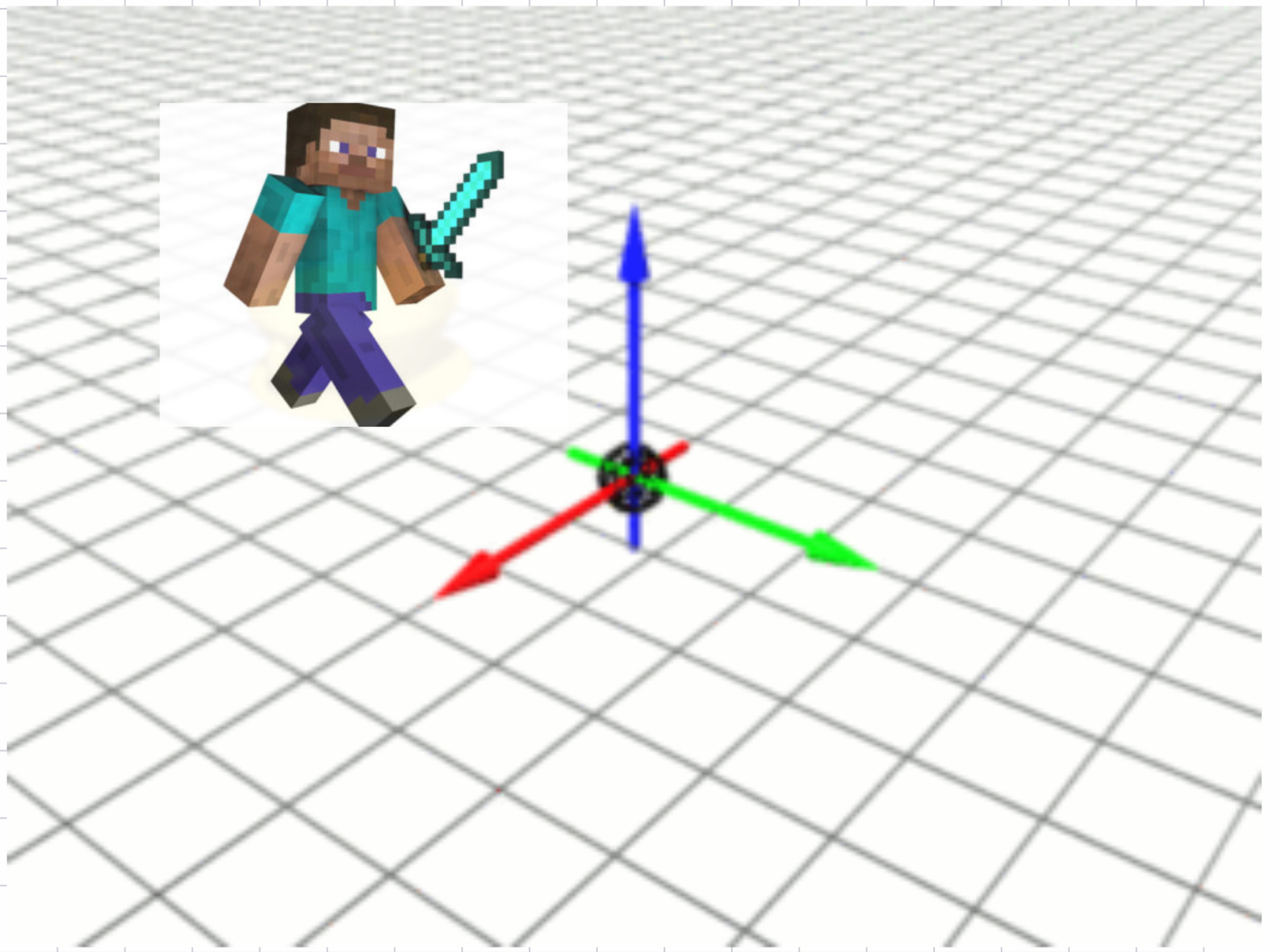
Announcements

- Work in groups - groups of 2 for MPs, groups of 2-4 for the final project.
- MP2 is out today, due Sep 29, 11:59pm.
- Reading: Chapter 6 and 7 of Shirley.
- Additional resource for geometric transformations background. Free online book: S. M. LaValle, "Planning Algorithms"

- Check piazza for the post on final project ideas. Start forming groups of 2-4.

Characterizing Object Motion





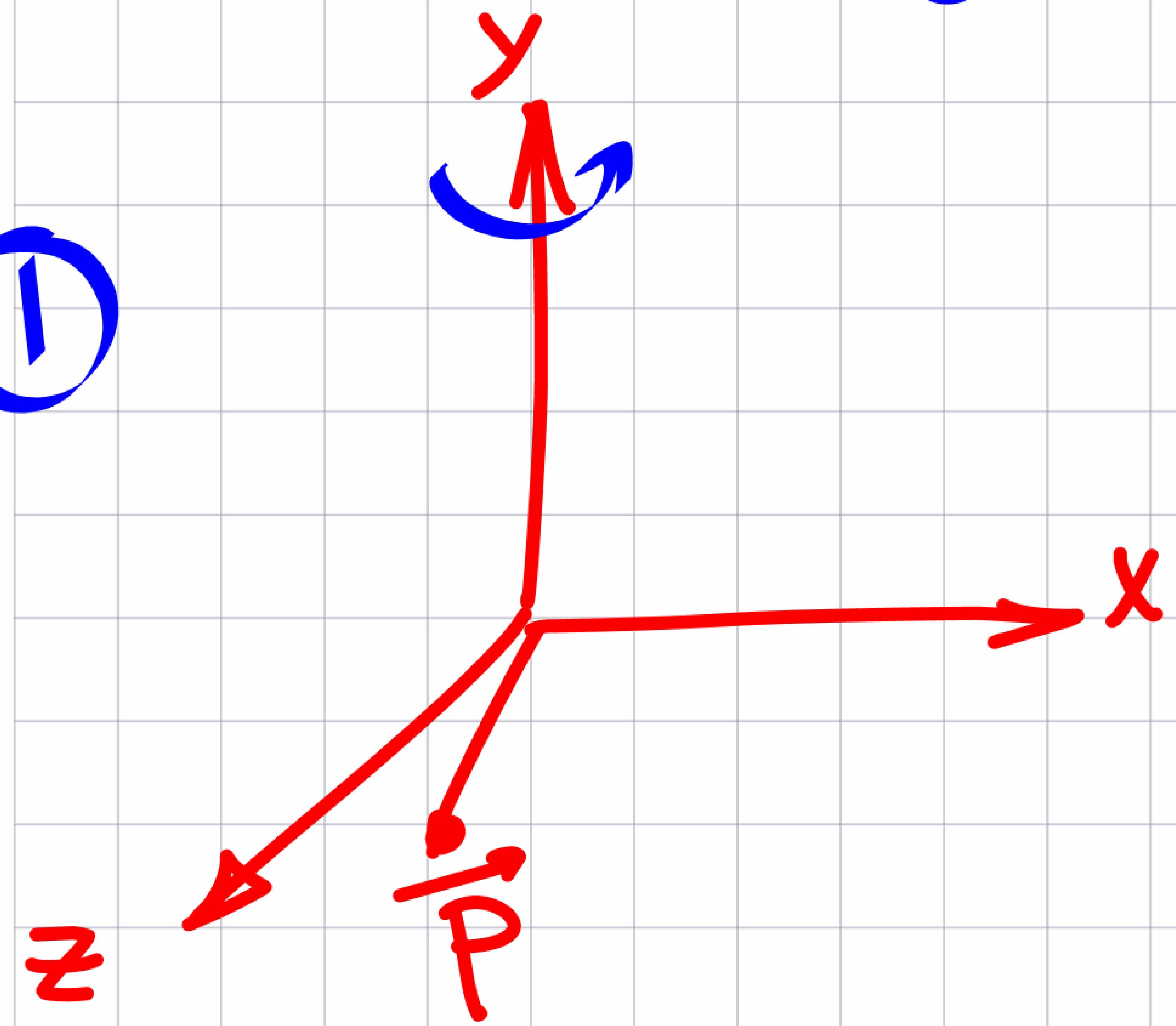
Chaining Matrices in Global Coordinate Frame

Steve is a Minecraft character. His head is a cube. The center of his head is the origin of the GLOBAL coordinate frame, in which his left pupil has coordinates (1, 0, 3).

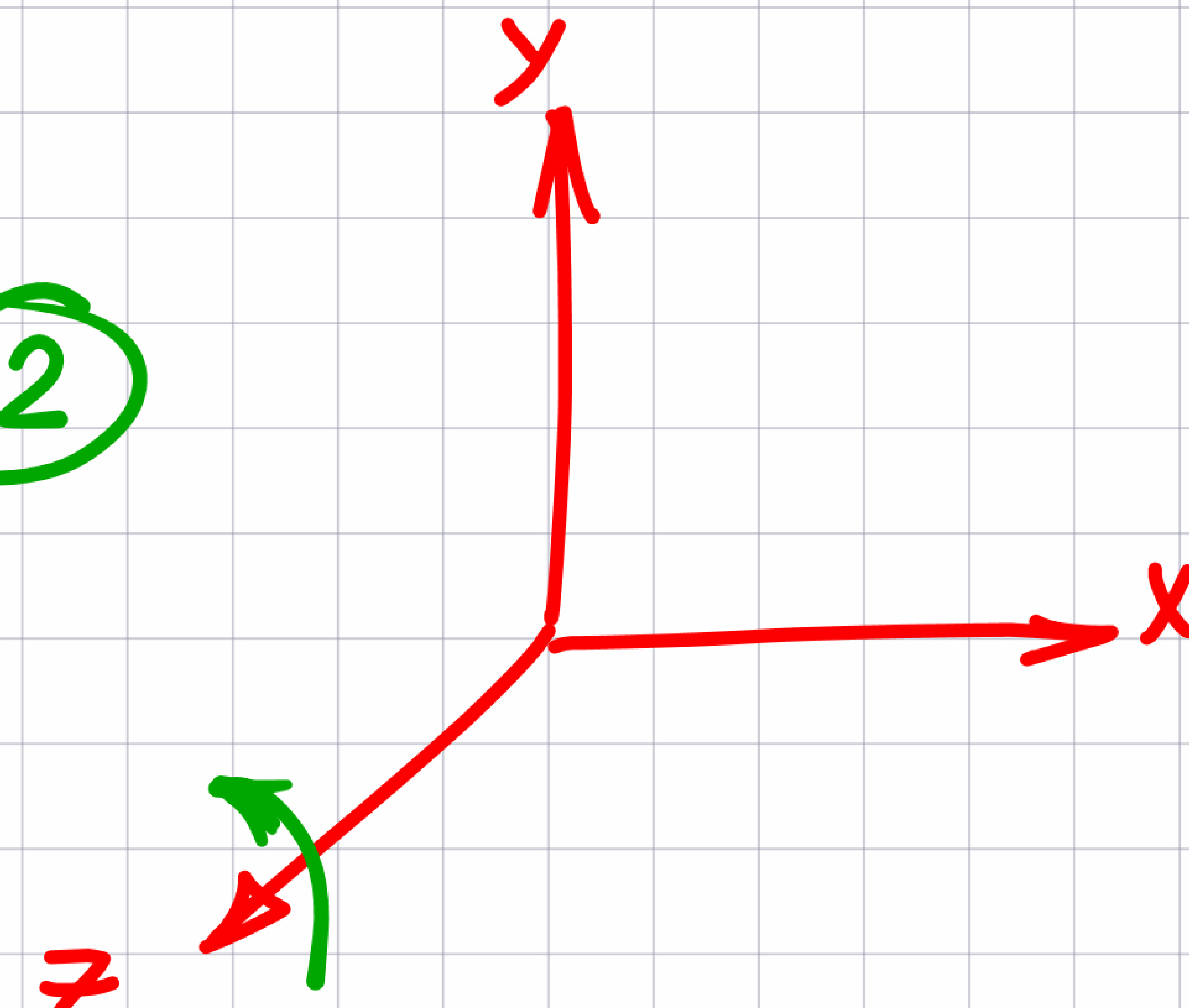
Calculate the coordinates of Steve's left pupil after Steve's head is turned first by a yaw of 90 degrees followed by a pitch by 90 degrees in GLOBAL coordinate frame.

Chaining Matrices in Global Coordinate Frame

1



2



Pupil: $\vec{p} = (1, 0, 3)$

Pupil: $p' = (\qquad)$

1 $p' = R_y\left(\frac{\pi}{2}\right) \cdot \vec{p} =$

$$= \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

2 $p'' =$

$$= R_z\left(\frac{\pi}{2}\right) \cdot p' =$$

$$= \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

Another Formulation of the Same Problem

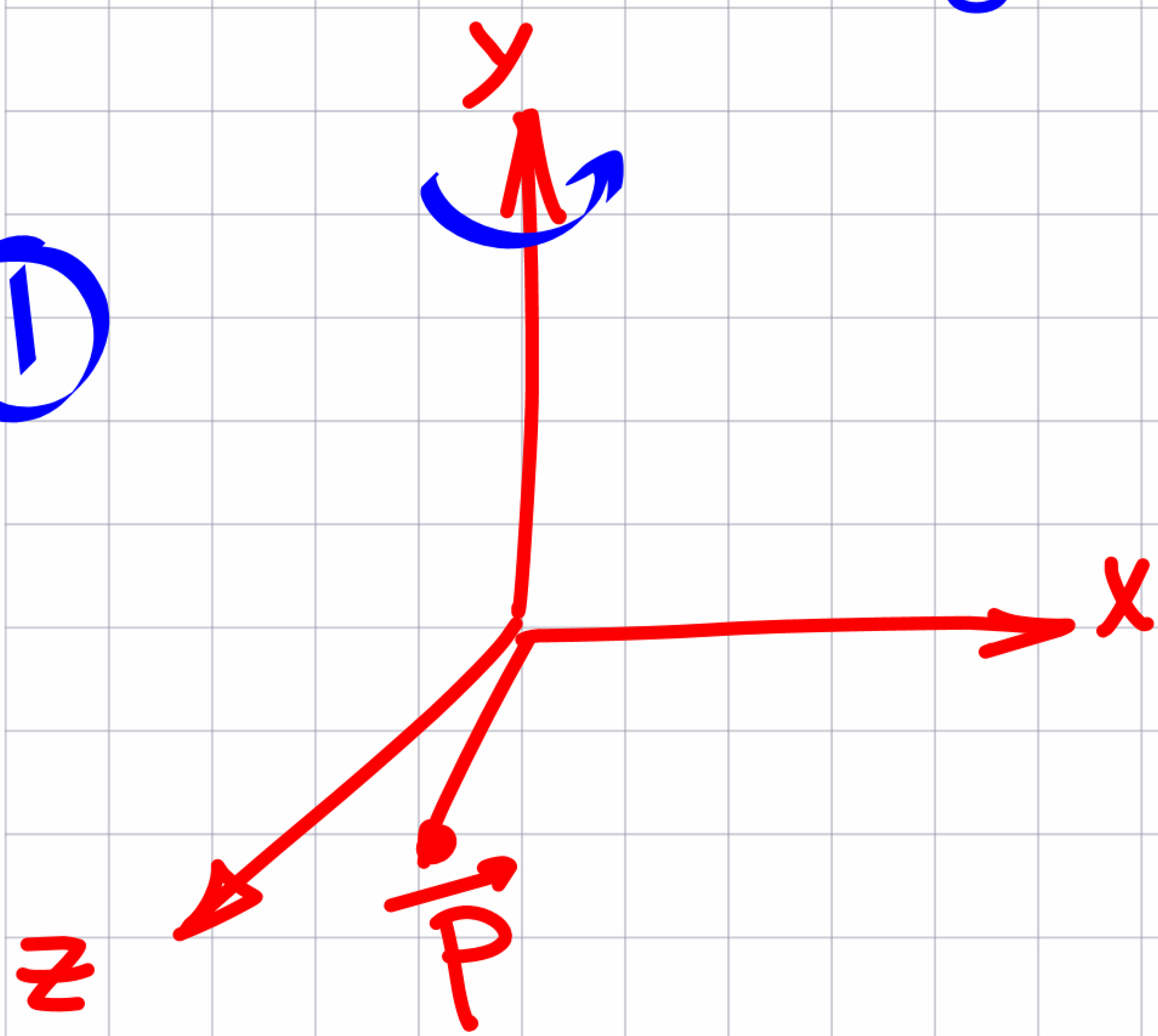
Steve is a Minecraft character. His head is a cube. Originally, his LOCAL coordinate frame coincides with the GLOBAL coordinate frame and his left pupil has coordinates (1, 0, 3).

Calculate the coordinates of Steve's left pupil after Steve turns his head first by a yaw of 90 degrees and then by a

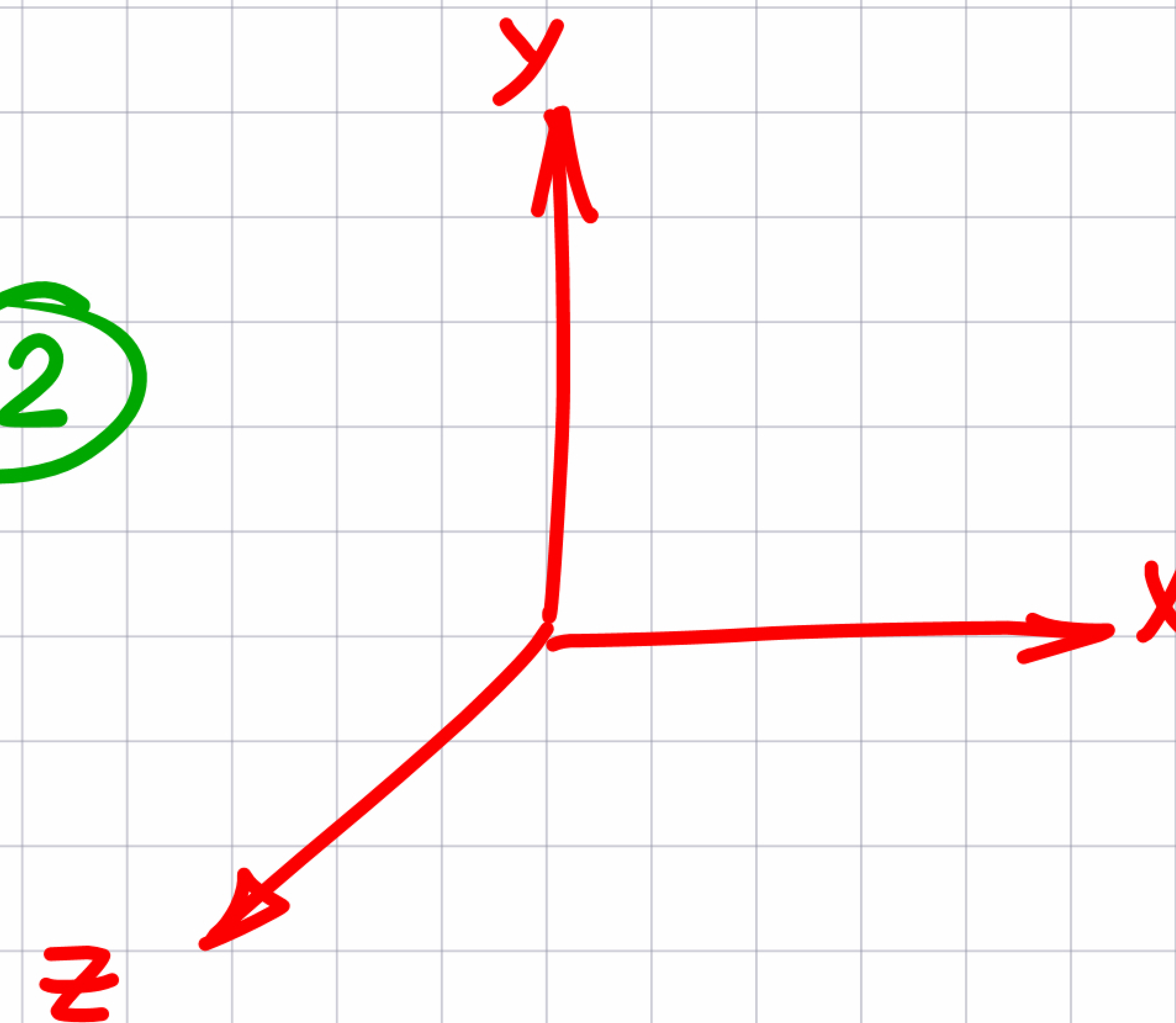
______________________________.

Chaining Matrices in Local Coordinate Frame

①



②



$$① \; R_y\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$② \; R_x\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} \; \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} \; \end{bmatrix}$$

Review: Matrix Associativity

$$A \cdot B \cdot C = A \cdot B \cdot C$$

$$R_1 \cdot R_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_1 \cdot R_2 \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Sanity Check: Using Quaternions Instead?

$$R_z\left(\frac{\pi}{2}\right) R_y\left(\frac{\pi}{2}\right) \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$R_y\left(\frac{\pi}{2}\right) R_x\left(-\frac{\pi}{2}\right) \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

Applying Quaternion Rotation to a Vector

Vector $(x, y, z) \in \mathbb{R}^3$

Rotate by quaternion q

$$p = (0, x, y, z)$$

$$p' = q \circ p \circ q^{-1}$$

To read the result take the last three components of p' only.

Limitations of 3x3 Matrices

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Transformations: Where are we?

Matrices

2x2

Rotations

Shear

Projection

Scale

Reflexion

3x3

Rotations

Yaw

Pitch

Roll

Shear

Projection

Scale

Reflexion

4x4

3D Rotations

Axis - Angle

Exponential coordinates

Quaternions

Translations using vectors

Chaining Translations and Rotations

Rotate by R, then translate by $t = (t_x, t_y, t_z)$

$$\begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Translate by $t = (t_x, t_y, t_z)$, then rotate by R

$$\begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Homogeneous Transformations: DOFs?

Rotate by R, then
translate by $t = (t_x, t_y, t_z)$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Homogeneous Transformation Matrix

Rotate by R, then translate by $t = (t_x, t_y, t_z)$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Have:

Want:

Homogeneous Transformation Matrices

Translate by $t = (t_x, t_y, t_z)$, then rotate by R

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} & & \\ & R & \\ & & \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix}$$

Homogeneous Transformation Matrix Inverse

Rotate by R, then translate by $t = (t_x, t_y, t_z)$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & t_x \\ \cdot & R & \cdot & t_y \\ \cdot & \cdot & \cdot & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Inverse:

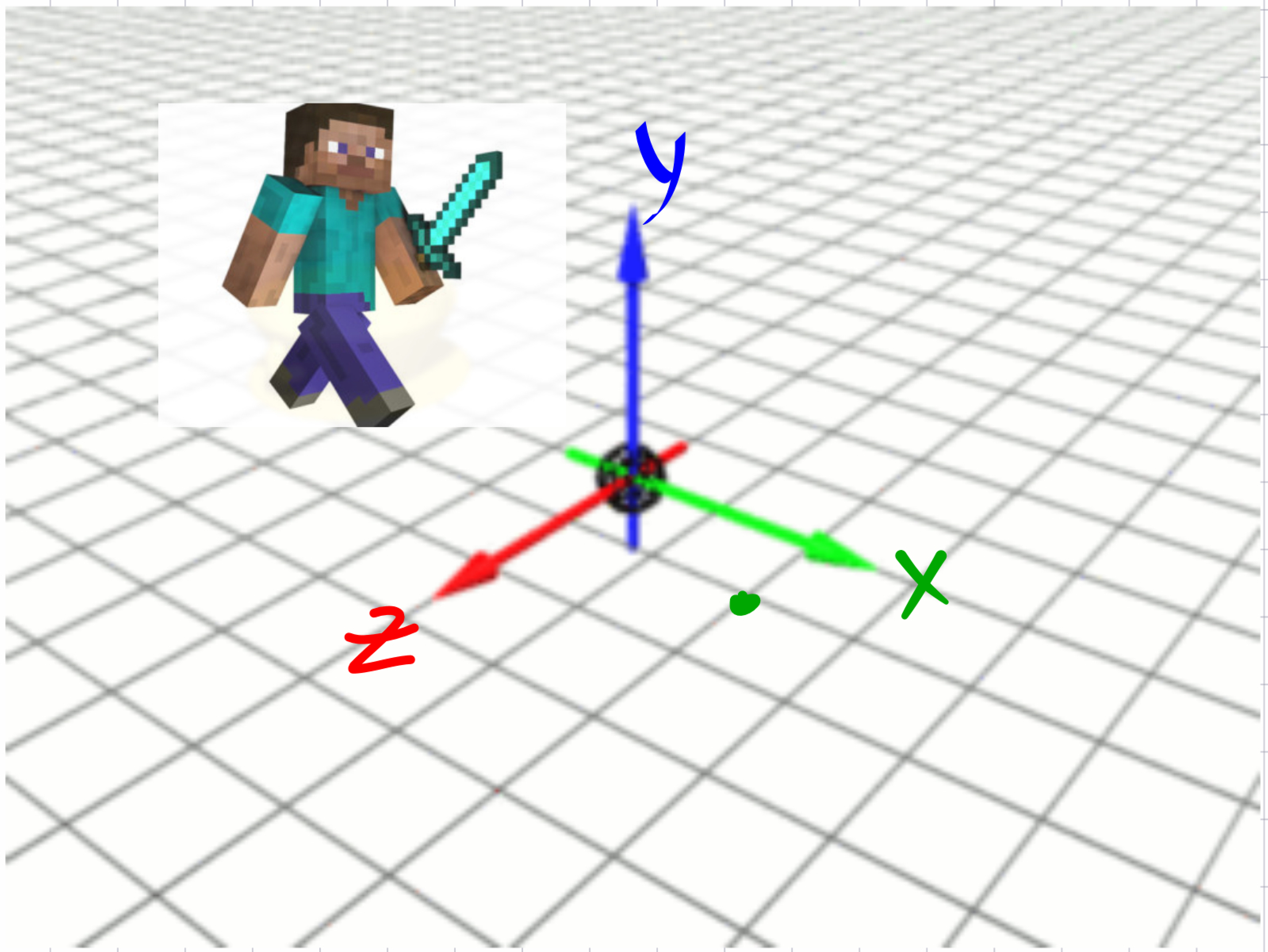
$$\begin{bmatrix} \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ & & & \end{bmatrix}$$

Homogeneous Transformation Matrix Inverse

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & 0 \\ \cdot & R & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cdot & \cdot & \cdot & 0 \\ \cdot & R & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

Switching Coordinate Frames



Flower: $\vec{x} = (2, 0, 1)$

Pupil: $\vec{p} = (1, 0, 3)$

Steve's transform matrix:

$$T = \begin{bmatrix} 0 & 0 & 1 & -10 \\ 0 & 1 & 0 & 10 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$