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2018-03-26.2 ←
cs579

cs 579: Computational Complexity: Lecture 18

4 6
NP complete
PH
$R^0(f) \geq \sqrt{n}$
D vs R tight

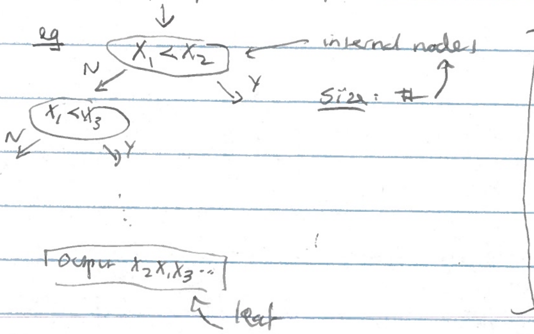
admin: ps 4 due
ps5 out
project topics released

today: query complexity

sorting: given n items $x_1, \ldots, x_n$, sort them
↑ abstract — have comparison oracle "$x_i < x_j$" [many algo fit in this model, but not all]

Recall: ↳ w/ $O(n \lg n)$ comparisons
↳ merge sort, heap sort, quicksort, etc

formalism: via decision tree — eg
||| query complexity

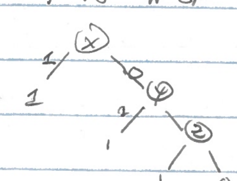


Prop: any sorting algo in comparison model requires $\Omega(n \lg n)$ depth

Pf: decision tree depth $d \to \underline{2^d \text{ leaves}}$ [binary answers]
all $n!$ orders must appear
$\Rightarrow d \geq \lg(n!) \geq \Omega(n \lg n)$

defn: a decision tree T on variables $x_1, \ldots, x_n$ is labelled binary tree w/ single root
- leaves $\in \{0,1\}$
- internal nodes labelled w/ var x_i, children "$x_i = 0$" "$x_i = 1$"
- depth = max distance from root to leaf
- computes $T: \{0,1\}^n \to \{0,1\}$ by — starting at root
  — if at "x_i" labelled node, follow "$x_i = b_i$" child, for b_i the value of x_i
- cost $(T, \bar{x})$ is # of variables queried when computing $T(\bar{x})$

eg: $OR_3(x, y, z) =$



cost $(T, 100) = 1$
$001 = 3$

$\Rightarrow$ cost $\leq$ depth
[per input] [worst case]

defn: $f: \{0,1\}^n \to \{0,1\}$ $D(f) = \min_{T \text{ computing } f} \underbrace{\max_x \text{cost}(T, x)}_{= \text{depth}(T)}$

Cor: $D(f) \leq n$ [only n variables to query]

Pf: query $x_1, x_2, \ldots, x_n$, output $f(\bar{x})$

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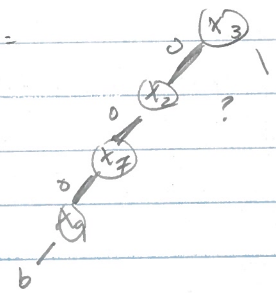
Prop: $D(OR_n) = n$

Pf: $\leq$: clear

$\geq$: use "adversary argument" on depth $< n$ tree

feed "$x_i = 0$" to each variable in tree until leaf is reached, output b

eg:



as depth $< n$, some variable x_{i_0} not queried

if $b=0$: set $x_{i_0} = 1 \Rightarrow OR_n(\bar{x}) = 1$
$T(\bar{x}) = 0$

$b = 1$ $x_i = 0$ all $i \Rightarrow OR_n = 0$
$T = 1$

def. $P^{dec} = \{ f : \{0,1\}^n \to \{0,1\}$ w/ $\underbrace{D(f) \leq \text{polylog}(n)}_{\text{"efficient"}} \}$ [families of functions]

$\Rightarrow OR \notin P^{dec}$

def: a b-certificate for $f : \{0,1\}^n \to \{0,1\}$ at x is a set $S \subseteq [n]$ and values $x|_S$

if $y|_S = x|_S \Rightarrow f(y) = f(x) = b$ [x at S certifies $f(x) = b$]

def: $N_b(f) =$ min k all x st $f(x) = b$ has b-certificate of size $\leq k$

[guess variables to query, then check]

ex: $N_1(OR_n) = 1$

$N_0(OR_n) = n$

def: $NP^{dec} = \{ f : \{0,1\}^n \to \{0,1\}$ w/ $N_1(f) \leq \text{polylog}(n) \}$

$coNP^{dec}$ N_0

lem: $D(f) \geq N_0(f), N_1(f)$

Pf. T decision tree for f, any x. $S_x := \{$queries of T on $x\}$

$\Rightarrow |S_x| \leq D(f)$. vars $\notin S$ don't affect T on $x \Rightarrow S_x$ is $f(x)$-cert

Cor: $P^{dec} \subseteq NP^{dec}, coNP^{dec}$

Cor: $P^{dec} \neq NP^{dec} \neq coNP^{dec}$ [expected]

$\not\ni$ $\ni$ $\not\ni$
OR_n

Prop: $D(f) \leq N_0(f) N_1(f)$

Cor: $P^{dec} = NP^{dec} \cap coNP^{dec}$ [not expected in turing machines]

Pf: $k = N_0(f)$ $\ell = N_1(f)$

$2^k \binom{n}{k}$ possible 0-cert $2^\ell \binom{n}{\ell}$ 1-cert

lem: $(S, x|_S)$ 0-cert & $(T, x|_T)$ 1-cert $\Rightarrow |S \cap T| > 0$

Pf: if $S \cap T = \emptyset$ then $\exists y$ $y|_S = x|_S \Rightarrow f(y) = 0$
$y|_T = x|_T$ $f(y) = 1$ $\Rightarrow \Leftarrow$


$x|_S$ x_T

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let $(S, x|_S)$ be a 0-certificate $|S| \le k$

query all variables in S get $y|_S$

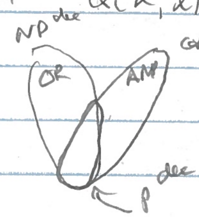
$y|_S = x|_S \Rightarrow f(y) = 0$

$\ne \Rightarrow$ all ℓ-certificates have $\le \ell - 1$ free vars

ie function on $[n] \setminus S$ has $N_0 \le k$ $N_1 \le \ell - 1$

recursion $Q(k, \ell) \le k + Q(k, \ell - 1) \le k \cdot \ell$

hence:



= Questions?

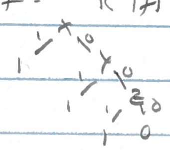
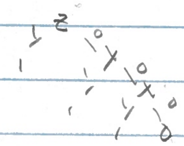
Q: randomization?

defn: let J be a probability distribution over decision trees $T_1, T_2, \ldots$ w/ each T_i computing f

$$\text{cost}(J) = \max_x \mathop{\mathbb{E}}_{T \sim J}[\text{cost}(T, x)]$$

$$R^0(f) = \min_{J \text{ computes } f} \text{cost}(J) \quad [[\text{ zero error }]]$$

$$ZPP^{dec} = \{f : R^0(f) \le \text{polylog}(n)\}$$

eg: $J = \frac{1}{2}$  $+ \frac{1}{2}$ 

cost			
100	1	3	2
010	2	2	2
001	3	1	2
000	3	3	3

Prop: $R^0(f) \ge N_0(f), N_1(f)$

Pf: $f(x) = b$ $\mathop{\mathbb{E}}_{T \sim J}[\text{cost}(T, x)] \le R^0(f)$

$\Rightarrow$ some T $\text{cost}(T, x) \le$ ... any T computes $f \Rightarrow$ b-cert size $\le R^0(f)$

Cor: $P^{dec} = ZPP^{dec}$

Fact: " $= BPP^{dec}$

Q: vs NP?

Prop: $R^0(OR) \ge \Omega(n)$

Cor: $NP^{dec} \ne ZPP^{dec} (= BPP^{dec})$

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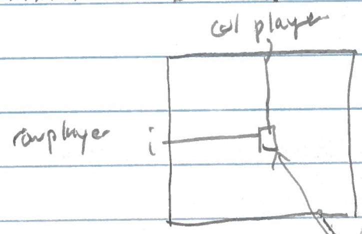
Thm [Von Neumann Minimax Thm]



A_{ij} = payoff to col = − payoff to row
zero sum game

row player goes first: $\min_i \max_j A_{ij}$

w/ randomness: $\min_R \max_j \mathbb{E}_{i \sim R} A_{ij}$ $=$ Pf: linear programming duality.

col player first: $\max_C \min_i \mathbb{E}_{j \sim C} A_{ij}$

Obs [Yao]: $R^0(f) = \underbrace{\min_{\mathcal{T}} \max_x \mathbb{E}_{T \sim \mathcal{T}} \mathrm{cost}(T,x)}_{\text{hard to lb}} = \max_{\mathcal{D}} \min_T \mathbb{E}_{x \sim \mathcal{D}} \mathrm{cost}(T,x)$

find $\mathcal{D}$ w/ large, lb ~ adversary method

eg for OR, $\mathcal{D}$ only uses hamming weight 0, 1 vecs

next time: communication complexity.