cs579: Computational Complexity Assigned: *Mon., Jan. 22, 2017*

Problem Set #1

Prof. Michael A. Forbes Due: *Mon., Feb. 5, 2017 (3:30pm)*

Some details from the syllabus.

- **Submission Policy:** Problem sets will be posted by 3:30pm on the day of release, and are due 3:30pm on the due date. An electronic (pdf) copy must be submitted by email to the course staff (`miforbes` and `rgandre2`). Each problem should be on a separate page. The subject line of the email should be "[cs579] ps$N$ submission" (N = pset number) and the filename must be "*NETID*_ps$N$.pdf" (N = pset number, *NETID* = your netid).
- **Collaboration Policy:** Students are forbidden from directly searching for solutions on the internet, but may consult the exercise-hints in Arora-Barak. That said, students are highly encouraged to collaborate in small groups. However, this must be a two-way collaboration. Students should not dictate complete solutions to other students, either verbally or written. Solutions must be written independently regardless of collaboration, and psets must list the collaborators you worked with.
- **Late Policy:** Students are highly encouraged to turn-in assignments on-time to avoid falling behind on the material, and to incentivize this any late homework will automatically lose 10%. However, to be flexible, students have a total of 6 late days (24 hours each, rounded up to an integral number of days), no more than 3 of which can be used for any particular homework. Problem sets will not be accepted for credit once the late days are exhausted.
- **Solutions:** Hard-copy sample solutions will be distributed to students when the problem sets are returned (please keep the internet free of easily-found solutions). These sample solutions will be selected from student submissions (with names omitted). Please inform the course staff if you wish to *opt-out* of ever being selected.

Each problem is worth 10 points.

1. (Arora-Barak Problem 1.15) Define a Turing Machine (TM) M to be *oblivious* if its head movements do not depend on the input but only on the input length. That is, M is oblivious if for every input $x \in \{0,1\}^\star$ and $i \in \mathbb{N}$, the location of M's head at the i-th step of execution on input x is only a function of $|x|$ and i. Show that for every time-constructible function $T : \mathbb{N} \to \mathbb{N}$, if $L \in \mathsf{TIME}(T(n))$ (on a one-tape TM), then there is an oblivious TM that decides L in time $O(T(n)^2)$ (that only uses two-tapes, an input tape and a work tape).
2. (Arora-Barak Problem 3.2) Show that $\mathsf{NP} \neq \mathsf{SPACE}(n^k)$, for all integers $k \geq 1$. Note that containments in either direction are unknown.
3. (Arora-Barak Problem 2.15) Let $G = (V, E)$ be an undirected graph. A *clique* in this graph is a subset $S \subseteq V$ such that all edges appear between vertices in S, that is, $\{(u, v) : u, v \in S\} \subseteq E$. An *independent set* is a set $S \subseteq V$ were *no* edges appear between vertices in S, that is $\{(u, v) : u, v \in S\} \subseteq (V \times V) \setminus E$. A *vertex cover* is a set $S \subseteq V$ where every edge $(u, v) \in E$ has either u or v in S. The CLIQUE language is $\{\langle G, k\rangle : G \text{ has a clique of size } k\}$; INDSET and $\mathsf{VERTEXCOVER}$ are defined analogously (however, note that the interesting case of vertex covers are small covers, while small cliques/independent-sets are easy to find.)

(a) Show that CLIQUE and INDSET are equivalent under polynomial-time many-one reductions.

(b) Show that $\mathsf{VERTEXCOVER}$ and INDSET are equivalent under polynomial-time many-one reductions.

(c) Using Theorem 2.15 from Arora-Barak, conclude that CLIQUE and $\mathsf{VERTEXCOVER}$ are NP-complete.

4. A non-deterministic time hierarchy theorem is proven in Arora-Barak (Theorem 3.2). We explore here an alternate proof. Let $f(n)$ be a time-constructible function. Let L be a language in $\mathsf{NTIME}(f(n))$. Taking the verifier perspective we can define L (for some deterministic Turing Machine M) as

$$L = \{x : \exists y \in \{0,1\}^{f(|x|)} \text{ so } M \text{ accepts } x \text{ in time } f(|x|) \text{ using non-determinism } y\} .$$

Now define language L' by

- $x = 1^i 0y$, $0 \leq |y| < f(i+1)$: $x \in L'$ iff $x0 \in L$ and $x1 \in L$.
- $x = 1^i 0y$, $|y| = f(i+1)$: $x \in L'$ iff M rejects input $1^i 0$ using non-determinism y.
- otherwise $x \notin L'$.

Show the following.

(a) $L \neq L'$.

(b) $L' \in \mathsf{NTIME}(g(n))$, for the smallest g you can prove.

(c) Modify the above construction to prove $\mathsf{NTIME}(f(n)) \subsetneq \mathsf{NTIME}(g(n))$ for the smallest g you can prove, and explain why a modification is needed.

Some hints.

2. Padding.