cs579: Computational Complexity Assigned: *Mon., Feb. 5, 2018*

Problem Set #2

Prof. Michael A. Forbes Due: *Mon., Feb. 19, 2018 (3:30pm)*

Some details from the syllabus (again).

1. **Submission Policy:** Problem sets will be posted by 3:30pm on the day of release, and are due 3:30pm on the due date. An electronic (pdf) copy must be submitted by email to the course staff (`miforbes` and `rgandre2`). Each problem should be on a separate page. The subject line of the email should be "[cs579] ps$N$ submission" (N = pset number) and the filename must be "*NETID*_ps$N$.pdf" (N = pset number, *NETID* = your netid).
2. **Collaboration Policy:** Students are forbidden from directly searching for solutions on the internet, but may consult the exercise-hints in Arora-Barak. That said, students are highly encouraged to collaborate in small groups. However, this must be a two-way collaboration. Students should not dictate complete solutions to other students, either verbally or written. Solutions must be written independently regardless of collaboration, and psets must list the collaborators you worked with.
3. **Late Policy:** Students are highly encouraged to turn-in assignments on-time to avoid falling behind on the material, and to incentivize this any late homework will automatically lose 10%. However, to be flexible, students have a total of 6 late days (24 hours each, rounded up to an integral number of days), no more than 3 of which can be used for any particular homework. Problem sets will not be accepted for credit once the late days are exhausted.
4. **Solutions:** Hard-copy sample solutions will be distributed to students when the problem sets are returned (please keep the internet free of easily-found solutions). These sample solutions will be selected from student submissions (with names omitted). Please inform the course staff if you wish to *opt-out* of ever being selected.

Each problem is worth 10 points.

1. Let $s(n)$ be a function which is space-constructible in $O(\log s(n))$-space. Let L be a language $L \in \mathsf{SPACE}(s(n))$.
   (a) Give a $O(\log s(n))$-space reduction $f : \{0,1\}^\star \to \{0,1\}^\star$ reducing $L \leq_l \mathsf{TQBF}$, and show that for any input x that $|f(x)| \leq O(s(|x|)^k)$, for the smallest constant k (independent of L) that you can establish.
   (b) Explain where L *does* impact the output-size of f.
   (c) Conclude that $\mathsf{TQBF} \notin \mathsf{SPACE}(n^\epsilon)$, for the largest constant $\epsilon > 0$ that you can establish.
2. (Arora-Barak Problem 6.3) Describe a decidable language in $\mathsf{P/poly}$ that is not in P.
3. Show that $\mathsf{TIME}\left(2^{n^{O(\lg n)}}\right) \not\subseteq \mathsf{P/poly}$.
4. (Arora-Barak Problem 4.5) Let $2\mathsf{SAT}$ be the language of satisfiable 2CNF formulas, such as $\varphi = (x \vee y) \wedge (\neg x \vee z) \wedge (w \vee w)$.

(a) Give a logspace-reduction from PATH to $\overline{2\mathsf{SAT}}$, the language of unsatisfiable 2CNF formulas.

(b) Show that $2\mathsf{SAT} \in \mathsf{NL}$.

(c) Conclude the $2\mathsf{SAT}$ is NL-complete under logspace-reductions.

Some hints.

4. Interpret a clause $(\neg x \vee y)$ as the implication $x \implies y$, or equivalently $\neg y \implies \neg x$. Create a graph where the vertices are the literals $\{x_i\}_i \cup \{\neg x_i\}_i$, and the edges are the implications derived from the 2CNF formula. Relate the existence of paths such as $x_i \rightsquigarrow \neg x_i$, to the satisfiability of the 2CNF formula.

Even more hints.

3. For each n, enumerate over all circuits of size $n^{\lg n}$ and find a language that disagrees with all of them.

4.(b) Show the following.

   i. Let $\varphi = \bigwedge(\ell_{i,1} \vee \ell_{i,2})$ be a 2CNF on n variables $x_1, \ldots, x_n$, where $\ell_{i,1}$ and $\ell_{i,2}$ are literals such as x_5 or $\neg x_3$. Create a graph $G = (V, E)$, where $V = \{x_i\}_i \cup \{\neg x_i\}_i$ and the edges are $E = \{\neg \ell_{i,1} \to \ell_{i,2}\}_i \cup \{\neg \ell_{i,2} \to \ell_{i,1}\}_i$.
   ii. Show that if there is some x where G contains the paths $x \rightsquigarrow \neg x$ and $\neg x \rightsquigarrow x$, then φ is unsatisfiable.
   iii. Show that if in G there is a path $x \rightsquigarrow y$ then there is a path $\neg y \rightsquigarrow \neg x$.
   iv. Show that if in G there is a path $x \rightsquigarrow y$ and a path $x \rightsquigarrow \neg y$ then there is a path $x \rightsquigarrow \neg x$.
   v. Show that if for every i, the paths $x_i \rightsquigarrow \neg x_i$ and $\neg x_i \rightsquigarrow x_i$ do not *both* appear in G, then φ is satisfiable. In particular, use the existence of such paths to decide a value for each x_i, and propagate this value by implication and show that no contradictions arise.
   vi. Use the above to design a coNL algorithm for $\mathsf{2SAT}$.