University of Illinois Fall 2017

ECE 313: Final Conflict Exam

Tuesday, December 19, 2017
1:30 p.m. — 4:30 p.m.

1. **[14 points]** Let X be a Gaussian random variable with mean 1 and variance 4. Let Y be a Gaussian random variable with mean 2 and variance 1. The correlation coefficient $\rho_{X,Y} = 1$.

   (a) What is $\mathrm{Cov}(X, Y)$?
   Solution: $\mathrm{Cov}(X, Y) = \rho\sigma_X\sigma_Y = 1 \times 2 \times 1 = 2$.

   (b) Suppose X and Y are jointly Gaussian. Find $P(2X + 3Y > 15)$. Express your answer in terms of the Q or Φ function.
   Solution:

$$E(2X + 3Y) = 2 \times 1 + 3 \times 2 = 8.$$

$$\mathrm{Var}(2X + 3Y) = 4\mathrm{Var}(X) + 9\mathrm{Var}(Y) + 12\mathrm{Cov}(X, Y) = 4 \times 4 + 9 + 12 \times 2 = 49.$$

$$P(2X + 3Y > 15) = P(Z > \frac{15 - 8}{7}) = Q(1).$$

2. **[22 points]** Consider a Poisson process with rate 1.

   (a) What is the probability that there are exactly two counts in the interval $[0, 2]$? Leave the final answer in terms of e.
   Solution: Let X be the number of counts in $[0, 2]$. X is Poisson with mean 2.

$$P(X = 2) = \frac{e^{-2}2^2}{2!} = 2e^{-2}$$

   (b) Given there are two counts in the interval $[0, 2]$, what is the probability that there are exactly three counts in the interval $[0, 4]$? Leave the final answer in terms of e.
   Solution: Let Y be the number of counts in $[0, 4]$. Y is Poisson with mean 4.

$$P(Y = 3|X = 2) = P(X = 1) = 2e^{-2}.$$

   (c) Given there are four counts in the interval $[0, 4]$, what is the probability that there are exactly two counts in the interval $[0, 2]$?
   Solution: Let Y be the number of users arriving in $[0, 4]$. Y is Poisson with mean 4.

$$P(X = 2|Y = 4) = \frac{P(X = 2)P(X = 2)}{P(Y = 4)} = \frac{4e^{-4}}{\frac{4^4e^{-4}}{4!}} = \frac{3}{8}.$$

3. **[16 points]** Consider orderings of the letters ILLINI.

   (a) How many orderings of the letters are there, if we do not distinguish the I's from each other and we do not distinguish the L's from each other?
   Solution: Since there are six letters, if the Is were labeled as I_1, I_2, I_3 and the Ls were labeled as L_1 and L_2; then the six letters would be distinct and there would be $6! = 720$ orderings, including $I_1I_3L_2NI_2L_1$. Each ordering without labels, such as $IILNIL$ corresponds to $3! \cdot 2 = 12$ orderings with labels, because there are 3! ways to label the three Is, and for each of those, there are two ways to label the Ls. Hence, there are $720/12 = 60$ ways to form a sequence of six letters, using three identical Is, two identical Ls, and one N.

(b) If the letters are randomly ordered, all orderings being equally likely, what is the probability the string NIL appears (in that order)?
Solution: From the previous part, we know the denominator for the probability computation will be 60. Now we need to find the size of the event that the string NIL appears. An example would be ILNILI. For this, we can group the letters NIL into a superletter X, so ILNILI is in one-to-one correspondence with ILXI. Now we want to see how many orderings there are of ILXI, and we see there are 4! orderings of four items, but we also need to correct for the fact there is a repetition of I, so divide by 2!. Thus, the size of the event is $4!/2! = 12$.
Hence the probability of the event is $12/60 = 1/5$.

4. [**22 points**] Alice is taking a test on basketball free throws. For each attempt, she succeeds or fails with equal probability. All attempts are independent.

Ten attempts are counted towards the result, starting from and including her first *successful* attempt, i.e., the consecutive misses at the beginning are not counted toward the ten.

(a) Find the mean of the number of successful attempts in Alice's test result.
Solution: Let X be the number of successful attempts. Then $X-1$ is Binomial$(9, 0.5)$. Hence $E[X] = 1 + 9 \cdot (0.5) = 5.5$.

(b) Find the mean of the total number of attempts (including the misses not counted toward the ten).
Solution: Let Y be the total number of attempts. Then $Y-9$ is Geom(0.5). Hence $E[Y] = 9 + \frac{1}{0.5} = 11$.

(c) Suppose Alice needs to have six successful attempts to pass the test. Find the probability she passes the test.
Solution: The probability she passes the test is $P\{X \geq 6\} = P\{X-1 \geq 5\} = p(5) + p(6) + p(7) + p(8) + p(9)$, where p is the pmf of a Binomial$(9, 0.5)$. Note $p(i) = p(9-i)$ for all i. Hence $P\{X \geq 6\} = 0.5$.

5. [**24 points**] Let the random variables X and Y have a joint PDF which is uniform over the triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$.

(a) Find the joint pdf of X and Y. Are X and Y independent?
Solution:

$$f_{XY}(u, v) = \begin{cases} 2 & 0 < u < 1, u + v < 1 \\ 0 & \text{otherwise} \end{cases}$$

X and Y are not independent because the support is not a product set.

(b) Find the marginal pdf of Y.
Solution: For $v \geq 1$ or $v \leq 0$, $f_Y(v) = 0$. For $0 < v < 1$,

$$\begin{aligned} f_Y(v) &= \int_{-\infty}^{\infty} f_{X,Y}(u, v) du = \int_0^{1-v} 2du \\ &= 2(1-v) \end{aligned}$$

(c) Find the conditional pdf of X given Y.
Solution: For $v \geq 1$ or $v \leq 0$, $f_{X|Y}(u|v)$ is not defined. For $0 < v < 1$,

$$\begin{aligned} f_{X|Y}(u|v) &= \frac{f_{X,Y}(u, v)}{f_Y(v)} \\ &= \begin{cases} \frac{1}{1-v} & \text{for } 0 < u < 1 - v \\ 0 & \text{else} \end{cases} \end{aligned}$$

(d) Find $E[X|Y = v]$.
Solution: The conditional probability density $f_{X|Y}(u|v) = 2/(1-v)$, for $0 < u < 1$ and $0 < v < 1-u$. When $v \geq 1$, or $v \leq 0$, $E[X|Y = v]$ is not defined. When $0 < v < 1$,

$$\begin{aligned} E[X|Y = v] &= \int_{-\infty}^{\infty} u f_{X|Y}(u|v) du \\ &= \int_0^{1-v} \frac{u}{1-v} du = \frac{1-v}{2} \end{aligned}$$

6. **[14 points]** Suppose we flip a fair coin n times. Let X be the number of heads in these trials.

(a) Bound $P\left(X \geq \frac{3}{4}n\right)$ using Markov's inequality.
Solution: By applying Markov's inequality, we have:

$$P\left(X \geq \frac{3}{4}n\right) \leq \frac{E[X]}{(3n/4)} = \frac{n/2}{3n/4} = \frac{2}{3}.$$

We observe that this bound is independent of n and it is reasonable to expect that this probability will go to 0 as n increases.

(b) Bound $P\left(X \geq \frac{3}{4}n\right)$ using Chebyshev's inequality.
Solution:

$$P\left(X \geq \frac{3}{4}n\right) = P\left(X - \frac{n}{2} \geq \frac{n}{4}\right) \leq P\left(|X - E[X]| \geq \frac{n}{4}\right) \leq \frac{n/4}{(n/4)^2} = \frac{4}{n}.$$

Clearly, this is a better bound than the previous since the probability goes to 0 as n increases.

7. **[14 points]** Let X and Y be independent random variables with pdf's

$$f_X(u) = \begin{cases} 1 & \text{if } 0 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

and

$$f_Y(v) = \begin{cases} 2v & \text{if } 0 \leq v \leq 1 \\ 0 & \text{else} \end{cases}$$

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(a) Find $E[X]$ and $E[Y]$.
Solution: $E[X] = \int_0^1 u du = 1/2$. $E[Y] = \int_0^1 v 2v dv = 2/3$.

(b) Let $Z = \max\{X, Y\}$. Find $E[Z]$.
Solution: The range of Z is $[0, 1]$. For $c \in [0, 1]$,

$$\begin{aligned} f_Z(c) &= f_X(c)F_Y(c) + F_X(c)f_Y(c) \\ &= 1 \cdot c^2 + c \cdot 2c \\ &= 3c^2. \end{aligned}$$

So $E[Z] = \int_0^1 c 3c^2 dc = 3/4$.

8. **[14 points]** Suppose we have data on the time x that a light bulb broke, and we assume this time is governed by an exponential random variable with unknown parameter λ.

(a) Find the maximum likelihood estimate for the failure rate function, $\hat{h}_{ML}(t)$ for all $t \geq 0$.
Solution: It is known that the failure rate function for an exponential random variable is just the constant parameter λ for all $t \geq 0$. Further, it is known that the maximum likelihood estimate for the parameter λ is just $1/x$. Hence, $\hat{h}_{ML}(t) = 1/x$ for all $t \geq 0$.

(b) Suppose we somehow have knowledge of λ for the parameter of the light bulb lifetime and we also have access to the realization u of a random variable U that is uniform in the interval $[0, 1]$. We want to simulate the lifetime of a light bulb that is independent and has an identical lifetime distribution as our light bulb by performing some function g on u. Find $g(\cdot)$.
Solution: The desired exponential distribution has support $\mathbb{R}^+$ and cdf F given by $F(c) = 1 - e^{-\lambda c}$ for $c \geq 0$ and $F(c) = 0$ for $c < 0$. By the known results for the probability integral transformation, we want $g(u) = F^{-1}(u)$. Since F is strictly and continuously increasing over the support, if $0 < u < 1$ then the value c of $F^{-1}(u)$ is such that $F(c) = u$. That is, we would like $1 - e^{-\lambda c} = u$ which is equivalent to $e^{-\lambda c} = 1 - u$, or $\lambda c = -\ln(1-u)$. Thus $c = -\frac{1}{\lambda}\ln(1-u)$. Therefore we can take $g(u) = -\frac{1}{\lambda}\ln(1-u)$ for $0 < u < 1$.

9. **[16 points]** Consider two Gaussian random variables: X with mean 0 and variance 4, and Y with mean 0 and variance 1.

(a) We see an observation w from one of the two Gaussian random variables. What is the ML decision rule to determine whether the observation came from X or from Y?
Solution: If we sketch out the two likelihood functions, we notice that Y is more peaked in the center, whereas X is wider in the extremes. Hence the ML decision rule that we desire will cut off an interval in the center that is symmetric around zero where we decide Y, and both tails will be where we decide X. We just need to find the boundaries of the region for Y, which we'll denote $\pm a$, and we focus on $+a$.
To find a, we just need to determine where the two pdfs intersect:

$$\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{a^2}{2}\right\} = \frac{1}{2\sqrt{2\pi}} \exp\left\{-\frac{a^2}{8}\right\}$$

This simplifies to:

$$\exp\left\{-\frac{a^2}{2}\right\} = \frac{1}{2} \exp\left\{-\frac{a^2}{8}\right\}$$

Taking ln of both sides yields

$$-\frac{a^2}{2} = -\ln 2 - \frac{a^2}{8}$$

Further simplifying,

$$\frac{3}{8}a^2 = \ln 2$$

then implies $a = \sqrt{\frac{8\ln 2}{3}}$. Thus we have our decision rule.

(b) Consider a decision rule that decides Y for the interval $[-a, a]$ and decides X for all other values in $\mathbb{R}$. Expressed in terms of the Q function, determine the probability of deciding X when we should have decided Y.
Solution: This is the probability that Y is either more than a or less than $-a$. Since Y is a standardized normal, so we can just use the basic Q function. By symmetry, we multiply one tail probability by two to get just $2Q(a)$.

10. **[14 points]** Given three random variables X, Y and Z. Find the linear MMSE estimator of $Z = aX + bY + c$. In particular, express a, b and c in terms of μ_X, μ_Y, μ_Z, σ_x, σ_Y, σ_z, and $\rho_{X,Y}$, $\rho_{X,Z}$, $\rho_{Y,Z}$.

    Solution: We want to minimize the MSE $E(aX+bY+c-Z)^2$. By finding the best constant estimator of $aX + bY - Z$, we have $c = a\mu_X + b\mu_Y - \mu_Z$. Substituting c and expand the terms, we have

    $$\begin{aligned} MSE &= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + \mathrm{Var}(X) \\ &= 2ab\mathrm{Cov}(X,Y) - 2a\mathrm{Cov}(X,Z) - 2b\mathrm{Cov}(Y,Z) \end{aligned}$$

    Differentiate with respect to a and b respectively, we obtain

    $$\begin{aligned} a\mathrm{Var}(X) + b\mathrm{Cov}(X,Y) &= \mathrm{Cov}(X,Z) \\ b\mathrm{Var}(Y) + a\mathrm{Cov}(X,Y) &= \mathrm{Cov}(Y,Z) \end{aligned}$$

    Solving for a and b, we obtain

    $$a = \frac{\sigma_z(\rho_{X,Z} - \rho_{X,Y}\rho_{Y,Z})}{\sigma_X(1-\rho^2_{X,Y})}$$

    $$b = \frac{\sigma_z(\rho_{Y,Z} - \rho_{X,Y}\rho_{Y,Z})}{\sigma_Y(1-\rho^2_{X,Y})}$$

11. **[30 points]** (3 points per answer)
    No partial credit is given for work shown. In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

    (a) Let E_1, E_2, E_3 be a partition of the sample space. An event A is independent of E_i, for $i = 1, 2, 3$, and $P(A) > 0$.

TRUE	FALSE	
☐	☐	Using Bayes' law, $P(E_i \mid A) = P(A)$.
☐	☐	The event A is independent of the event $(E_1 \bigcup E_2)$.

    Solution: False,True

    (b) Let X_n be Binomial(n, p) for $n = 1, 2, \ldots$.

TRUE	FALSE	
☐	☐	Let $p = \frac{1}{n}$ and let $n \to \infty$. The limiting distribution of X_n is discrete.
☐	☐	Fix p and let $n \to \infty$. The limiting distribution of $\frac{X_n - np}{\sqrt{n}}$ is discrete.

    Solution: True,False

    (c) Suppose $X_1, ..., X_n$ are random variables drawn from the same distribution with mean μ. Let $\hat{\mu}$ be the sample mean and $\rho(X_i, X_j)$ be the correlation coefficient between any two random variables X_i, X_j. Suppose that X_1 is first drawn independently of everything else.

TRUE	FALSE	
☐	☐	Let $\rho(X_i, X_{i+1}) = -1$ for $i = 1, 2, \ldots, n-1$. Then $\hat{\mu} \to \mu$.
☐	☐	Let $\rho(X_i, X_j) = 0$ for $i \neq j$. Then $\lim_{n\to\infty} \hat{\mu} \neq \mu$.

    Solution: False,False

(d) Suppose that θ is estimated by observing the random variable X using the ML, MMSE, linear MMSE estimators $\hat{\theta}_{ML}, E[\theta|X], \hat{E}[\theta|X]$ respectively. Let $\text{MSE}(\theta, \hat{\theta})$ be the mean square error of each estimator with respect to the true θ, and $\mathcal{L}(X; \hat{\theta})$ be the value of the likelihood function when we plug in the values of X and $\hat{\theta}$.

TRUE	FALSE	
☐	☐	$\text{MSE}(\theta, E[\theta|X]) \leq \text{MSE}(\theta, \hat{E}[\theta|X])$ and $\text{MSE}(\theta, E[\theta|X]) \leq \text{MSE}(\theta, \hat{\theta}_{ML})$.
☐	☐	$\text{MSE}(\theta, \hat{E}[\theta|X]) \leq \text{MSE}(\theta, \hat{\theta}_{ML})$.
☐	☐	$\mathcal{L}(X; \hat{\theta}_{ML}) \geq \mathcal{L}(X; E[\theta|X])$ and $\mathcal{L}(X; \hat{\theta}_{ML}) \geq \mathcal{L}(X; \hat{E}[\theta|X])$.
☐	☐	$\mathcal{L}(X; E[\theta|X]) \leq \mathcal{L}(X; \hat{E}[\theta|X])$.

Solution: True, False, True, False