University of Illinois Fall 2017

ECE 313: Final Exam

Monday, December 18, 2017
1:30 p.m. — 4:30 p.m.

1. **[14 points]** Suppose Y and W are jointly Gaussian random variables with $E[Y] = 2$, $E[W] = 0$, $\mathrm{Var}(Y) = 1$, $\mathrm{Var}(W) = 9$, and $\rho_{Y,W} = 0.25$. Let $X = 2Y + W$.

   (a) Find $E[X]$ and $\mathrm{Var}(X)$.
   Solution:
   $$E[X] = 2E[Y] + E[W] = 4$$
   $$\mathrm{Var}(X) = \mathrm{Cov}(2Y+W, 2Y+W) = 4\mathrm{Var}(Y)+\mathrm{Var}(W)+4\mathrm{Cov}(Y,W) = 4+9+4\times 0.25\times 1\times 3 = 16$$

   (b) Calculate $P(X \geq 20)$. Express your answer in terms of the Q or Φ function.
   Solution: X is a Gaussian random variable with mean 4 and variance 16, therefore
   $$P(X \geq 20) = P\left(\frac{X-4}{4} \geq 4\right) = Q(4) = 1 - \Phi(4)$$

2. **[16 points]** Consider a binary hypothesis testing problem based on observation of n independent tosses of a coin. Let $X_1, \ldots, X_n$ denote the outcomes, $X_i = 1$ for head and $X_i = 0$ for tail. The two hypotheses are:
   $$H_0: \text{ for each flip, head shows with } p \leq \frac{1}{2}$$
   $$H_1: \text{ for each flip, head shows with } q > \frac{1}{2}$$

   (a) What are the probabilities $p_i(k)$ of having k heads given the hypothesis H_i, $i = 1, 2$?
   Solution:
   $$p_0(k) = \binom{n}{k} p^k (1-p)^{n-k}$$
   $$p_1(k) = \binom{n}{k} q^k (1-q)^{n-k}$$

   (b) Find the expression for the likelihood ratio. (Keep the results in terms of n, k, p, and q.)
   Solution:
   $$\Lambda = \frac{p_1(k)}{p_0(k)} = \left(\frac{q}{p}\right)^k \left(\frac{1-q}{1-p}\right)^{n-k}$$
   If $\Lambda > 1$, ML rule selects H_1 and if $\Lambda < 1$, ML rule selects H_0

   (c) In particular, if $n = 100$ $k = 51$, $p = 1/3$ and $q = 2/3$, which hypothesis does the maximum likelihood decision rule select?
   Solution:
   $$\Lambda = \frac{\left(\frac{2}{3}\right)^{51}\left(\frac{1}{3}\right)^{49}}{\left(\frac{1}{3}\right)^{51}\left(\frac{2}{3}\right)^{49}} = 4 > 1.$$
   Therefore, ML decision rule selects H_1.

3. **[22 points]** Consider a Poisson process with rate 1.

   (a) What is the probability that there are exactly two counts in the interval $[0, 2]$? Leave the final answer in terms of e.
   Solution: Let X be the number of counts in $[0, 2]$. X is Poisson with mean 2.

$$P(X = 2) = \frac{e^{-2}2^2}{2!} = 2e^{-2}$$

   (b) Given there are two counts in the interval $[0, 2]$, what is the probability that there are exactly three counts in the interval $[0, 4]$? Leave the final answer in terms of e.
   Solution: Let Y be the number of counts in $[0, 4]$. Y is Poisson with mean 4.

$$P(Y = 3|X = 2) = P(X = 1) = 2e^{-2}.$$

   (c) Given there are four counts in the interval $[0, 4]$, what is the probability that there are exactly two counts in the interval $[0, 2]$?
   Solution: Let Y be the number of users arriving in $[0, 4]$. Y is Poisson with mean 4.

$$P(X = 2|Y = 4) = \frac{P(X = 2)P(X = 2)}{P(Y = 4)} = \frac{4e^{-4}}{\frac{4^4 e^{-4}}{4!}} = \frac{3}{8}.$$

4. **[22 points]** Let X and Y be two Gaussian random variables with mean 0 and variance 1.

   (a) If $\rho_{X,Y} = 0.5$, what is the minimum MSE linear estimator of Y given $X = 10$?
   Solution: $\hat{E}(Y|X) = \rho X = 5$.

   (b) If X and Y are independent, what is the minimum MSE estimator of Y given $X = 10$?
   Solution: $E(Y|X) = E(Y) = 0$.

   (c) If X and Y are jointly Gaussian and $\rho_{X,Y} = 0.2$, what is the conditional pdf of Y given $X = 5$?
   Solution: Since X and Y are jointly Gaussian, $f_{Y|X}(v|u)$ is Gaussian with mean ρX and variance $1 - \rho^2$. Hence $\mathcal{N}(1, 0.96)$

5. **[16 points]** Consider a school where three courses are assigned to a student, chosen uniformly at random from six possible courses, $\{A, B, C, D, E, F\}$. In order to graduate, the student must complete course A, one out of $\{B, C\}$, and one out of $\{D, E, F\}$.

   (a) What is the probability that the student gets courses that lead to graduation?
   Solution: All $\binom{6}{3} = 20$ unordered sets of courses are equally likely. Of these, the following ones would lead to graduation: $(ABD, ABE, ABF, ACD, ACE, ACF)$, of which there are six. Hence the probability is $6/20 = 3/10$.

   (b) The student likes course B. Given that the student graduates, what is the probability that the student gets the course she likes?
   Solution: If the student graduates, she has received either course B or course C. These are equally likely, so $1/2$.

6. **[22 points]** Alice is taking a test on basketball free throws. For each attempt, she succeeds or fails with equal probability. All attempts are independent.

   Ten attempts are counted towards the result, starting from and including her first *successful* attempt, i.e., the consecutive misses at the beginning are not counted toward the ten.

(a) Find the mean of the number of successful attempts in Alice's test result.
Solution: Let X be the number of successful attempts. Then $X - 1$ is Binomial$(9, 0.5)$. Hence $E[X] = 1 + 9 \cdot (0.5) = 5.5$.

(b) Find the mean of the total number of attempts (including the misses not counted toward the ten).
Solution: Let Y be the total number of attempts. Then $Y - 9$ is Geom(0.5). Hence $E[Y] = 9 + \frac{1}{0.5} = 11$.

(c) Suppose Alice needs to have six successful attempts to pass the test. Find the probability she passes the test.
Solution: The probability she passes the test is $P\{X \geq 6\} = P\{X - 1 \geq 5\} = p(5) + p(6) + p(7) + p(8) + p(9)$, where p is the pmf of a Binomial$(9, 0.5)$. Note $p(i) = p(9 - i)$ for all i. Hence $P\{X \geq 6\} = 0.5$.

7. **[14 points]** Suppose we flip a fair coin n times. Let X be the number of heads in these trials.

(a) Bound $P\left(X \geq \frac{3}{4}n\right)$ using Markov's inequality.
Solution: By applying Markov's inequality, we have:

$$P\left(X \geq \frac{3}{4}n\right) \leq \frac{E[X]}{(3n/4)} = \frac{n/2}{3n/4} = \frac{2}{3}.$$

We observe that this bound is independent of n and it is reasonable to expect that this probability will go to 0 as n increases.

(b) Bound $P\left(X \geq \frac{3}{4}n\right)$ using Chebyshev's inequality.
Solution:

$$P\left(X \geq \frac{3}{4}n\right) = P\left(X - \frac{n}{2} \geq \frac{n}{4}\right) \leq P\left(|X - E[X]| \geq \frac{n}{4}\right) \leq \frac{n/4}{(n/4)^2} = \frac{4}{n}.$$

Clearly, this is a better bound than the previous since the probability goes to 0 as n increases.

8. **[10 points]** Assume that $X_1, X_2, \ldots, X_n$ are i.i.d. data drawn from Unif$[a, b]$. Find the Maximum Likelihood estimates for a and b.

Solution: The likelihood function for this problem is given by:

$$\mathcal{L}(X_1, X_2, \ldots, X_n; a, b) = \frac{1}{(b-a)^n}.$$

The MLEs for a, b are obtained by the maximizing values of the likelihood under the restriction that the data have to be compatible with the interval $[a, b]$ (i.e., all data points have to fall in this interval). These considerations show that

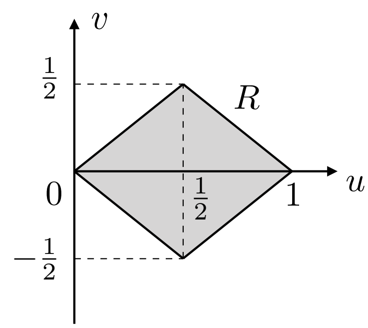
$$\hat{a} = \min\{X_1, X_2, \ldots, X_n\}, \quad \hat{b} = \max\{X_1, X_2, \ldots, X_n\},$$

which jointly maximize the likelihood.

9. **[18 points]** Suppose X and Y have the following joint pdf $f_{X,Y}$.

$$f_{X,Y}(u, v) = \begin{cases} C(u - v) & \text{if } (u, v) \text{ is in the region } R \\ 0 & \text{else} \end{cases},$$

where the constant C is to be determined and the region R is shown in the figure.



Let $W = X + Y$ and $Z = X - Y$.

(a) Find C.
Solution:

$$\begin{aligned}\int\int f_{X,Y}(u,v)dudv &= C\left(\int_0^{1/2}\int_{-u}^{u}(u-v)dvdu + \int_{1/2}^{1}\int_{u-1}^{1-u}(u-v)dvdu\right)\\ &= C\left(\int_0^{1/2}2u^2du + \int_{1/2}^{1}2u(1-u)du\right)\\ &= \frac{C}{4}.\end{aligned}$$

So $C = 4$.

(b) Find $f_{W,Z}$ (you can express it in terms of C). Are W and Z independent?
Solution: $\begin{pmatrix} W \\ Z \end{pmatrix} = A\begin{pmatrix} X \\ Y \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. By the linear transformation,

$$\begin{aligned}f_{W,Z}(\alpha,\beta) &= \frac{1}{|\det A|}f_{X,Y}\left(A^{-1}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}\right)\\ &= \frac{1}{2}f_{X,Y}\left(\frac{\alpha+\beta}{2},\frac{\alpha-\beta}{2}\right)\\ &= \begin{cases} 2\beta & \text{if } \left(\frac{\alpha+\beta}{2},\frac{\alpha-\beta}{2}\right)\in R \\ 0 & \text{else}\end{cases}\\ &= \begin{cases} 2\beta & \text{if } 0\le\alpha\le 1, 0\le\beta\le 1 \\ 0 & \text{else}\end{cases}.\end{aligned}$$

The marginal pdf's are

$$f_W(\alpha) = \begin{cases} 1 & \text{if } 0\le\alpha\le 1 \\ 0 & \text{else}\end{cases}$$

and

$$f_Z(\beta) = \begin{cases} 2\beta & \text{if } 0\le\beta\le 1 \\ 0 & \text{else}\end{cases}.$$

Since the support is a product set and $f_{W,Z}(\alpha,\beta) = f_W(\alpha)f_Z(\beta)$, W and Z are independent.

10. **[16 points]** Let $X_1, X_2, \ldots, X_n$ be i.i.d. data drawn from Unif$[0,\theta]$. Assume that $Y_n = \max\{X_1, X_2, \ldots, X_n\}$.

(a) Consider the interval estimator for θ, $[aY_n, bY_n], 1 \le a < b$. Find $P(\theta \in [aY_n, bY_n])$.
Solution: We first compute the CDF of Y_n:

$$F_{Y_n}(y) = (F_X(y))^n = \frac{1}{\theta^n} y^n.$$

therefore, the pdf of Y_n is given by:

$$f_{Y_n}(y) = \frac{1}{\theta^n} n y^{n-1}, \quad y \in [0, \theta].$$

We can now compute:

$$\begin{aligned} P(\theta \in [aY_n, bY_n]) &= P\left(aY_n \le \theta \le bY_n\right) = P\left(\frac{\theta}{b} \le y \le \frac{\theta}{a}\right) \\ &= \frac{1}{\theta^n} \int_{\theta/b}^{\theta/a} n y^{n-1} dy = \frac{1}{\theta^n}\left[\left(\frac{\theta}{a}\right)^n - \left(\frac{\theta}{b}\right)^n\right] \\ &= \left(\frac{1}{a}\right)^n - \left(\frac{1}{b}\right)^n. \end{aligned}$$

The obtained probability is clearly independent of θ.

(b) Repeat the calculation for $P(\theta \in [Y_n + c, Y_n + d]), 0 < c < d$ and compute $\lim_{\theta \to \infty} P(\theta \in [Y_n + c, Y_n + d])$.
Solution:

$$\begin{aligned} P(\theta \in [Y_n + c, Y_n + d]) &= P(\theta - d \le y \le \theta - c) = \frac{1}{\theta^n} \int_{\theta-d}^{\theta-c} n y^{n-1} dy \\ &= \frac{1}{\theta^n}\left[(\theta - c)^n - (\theta - d)^n\right] \end{aligned}$$

and therefore

$$\lim_{\theta \to \infty} \frac{1}{\theta^n}\left[(\theta - c)^n - (\theta - d)^n\right] = 0.$$

11. **[30 points]** (3 points per answer)
No partial credit is given for work shown. In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Let X_n be Binomial(n, p) for $n = 1, 2, \ldots$.

TRUE	FALSE	
☐	☐	Let $p = \frac{1}{n}$ and let $n \to \infty$. The limiting distribution of X_n is discrete.
☐	☐	Fix p and let $n \to \infty$. The limiting distribution of $\frac{X_n - np}{\sqrt{n}}$ is discrete.

Solution: True,False

(b) Given three random variables X, Y, and Z. Suppose X and Y are uncorrelated. Also suppose Y and Z are uncorrelated.

TRUE	FALSE	
☐	☐	X, Y, and Z are uncorrelated.
☐	☐	$E[XY] = 0$.

Solution: False,False

(c) Suppose $X_1, ..., X_n$ are i.i.d. random variables, with mean μ and variance σ^2. Let $\hat{\mu}$ be the sample mean.

TRUE	FALSE	
☐	☐	$\frac{1}{n-1}\sum_{k=1}^{n}(X_k - \hat{\mu})^2$ is an unbiased estimator of σ^2.
☐	☐	$\frac{1}{n}\sum_{k=1}^{n}(X_k - \mu)^2$ is an unbiased estimator of σ^2.

Solution: True,True

(d) Consider a binary hypothesis testing problem with $H_0 : X \sim \mathcal{N}(0, 1)$ and $H_1 : X \sim \mathcal{N}(1, 1)$.

TRUE	FALSE	
☐	☐	If the ML decision rule is employed, then $P_{FA} = P_{miss}$
☐	☐	For the MAP decision rule, $P_e = P_{FA}P(H_1) + P_{miss}P(H_0)$

Solution: True, False

(e) Let E_1, E_2, E_3 be a partition of the sample space. An event A is independent of E_i, for $i = 1, 2, 3$, and $P(A) > 0$.

TRUE	FALSE	
☐	☐	Using Bayes' law, $P(E_i|A) = P(A)$.
☐	☐	The event A is independent of the event $(E_1 \bigcup E_2)$.

Solution: False,True