ECE 313: Lecture 13
Confidence intervals

Recall: Throw a coin n times. Observe $X = k$ 'H'

$\{ H, T \}$ prob p, $1-p$

$X \sim \text{Binomial}(n, p)$

Ex: $n = 10$; $X = k = 3$ $\Rightarrow$ $\hat{p}_{ML} = \frac{k}{n} = 0.3$

$n = 100$; $k = 30$ $\Rightarrow$ $\hat{p}_{ML} = 0.3$

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n = 10
p = 3/10
pmf_b = binomial(n, p)
plt.stem(pmf_b)
plt.title('pmf of Binomial($n, p$) for $n = {}, p = {}$'.format(n, p));
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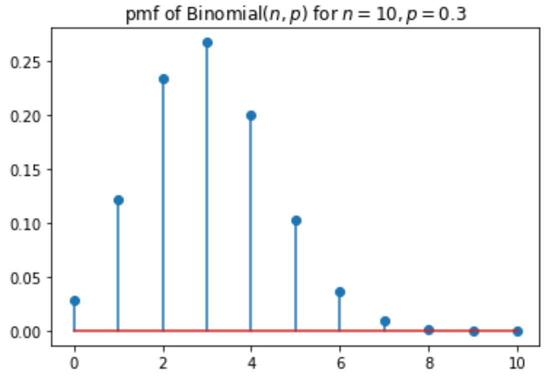
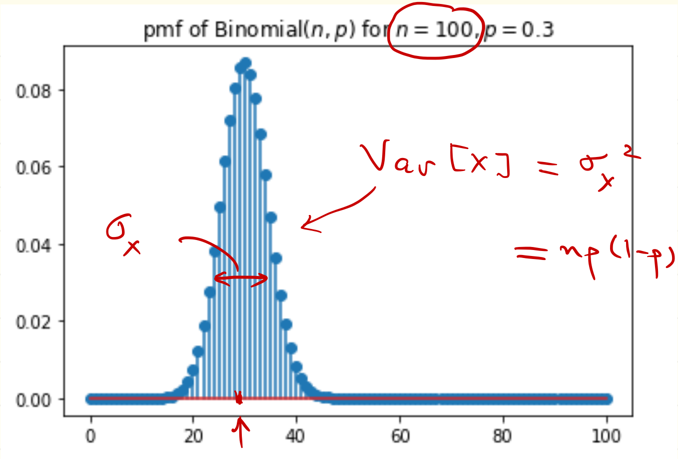


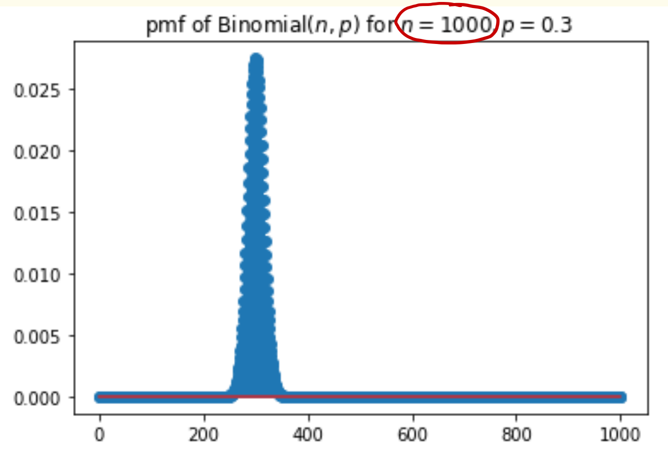


σ_x

$Var[x] = \sigma_x^2$

$= np(1-p)$

$E[x] = 30 = p.n$



$$E\left[\frac{X}{n}\right] = p$$

$$Var\left[\frac{X}{n}\right] = \frac{Var[X]}{n^2}$$

$$= \frac{np(1-p)}{n^2}$$

$$= \frac{p(1-p)}{n}$$

Recall: Chebychev inequality

$$\begin{cases} X : \text{r.v.} \\ E[X] = \mu \\ Var[X] = \sigma_X^2 \end{cases}$$

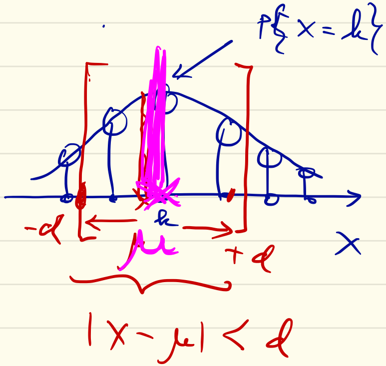
$\boxed{d = a\sigma}$

$$P[\,|X-\mu| \geq d\,] \leq \frac{\sigma_X^2}{d^2}$$

$$\Longleftrightarrow \quad P\{|X-\mu| < d\} \geq 1 - \frac{\sigma_X^2}{d^2}$$

$$\Longleftrightarrow \quad P\{\mu - d < X < \mu + d\} \geq 1 - \underbrace{\frac{\sigma_X^2}{d^2}}_{a^2}$$

$$\Longleftrightarrow \quad P\{|X-\mu| < \underbrace{a\sigma}_{d}\} \geq 1 - \frac{1}{a^2}$$



Ex: $X \sim \text{Binom}(n, p)$ — unknown parameter

$$\begin{cases} E\left[\frac{X}{n}\right] = \frac{E[X]}{n} = p \\ Var\left[\frac{X}{n}\right] = \frac{Var[X]}{n^2} = \frac{p(1-p)}{n} \end{cases}$$

Observe $X = k$

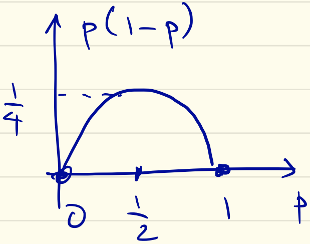
$$\Rightarrow \quad \hat{p} = \frac{k}{n} = \frac{X}{n}$$

From Chebychev:

$$P\left\{\left|\underbrace{\frac{X}{n}}_{\hat{p}} - p\right| < a\sqrt{\frac{p(1-p)}{n}}\right\} \geq 1 - \frac{1}{a^2}$$

$$\Rightarrow \quad \mathbb{P}\left\{\hat{p} - a\sqrt{\frac{p(1-p)}{n}} < p < \hat{p} + a\sqrt{\frac{p(1-p)}{n}}\right\} > 1 - \frac{1}{a^2}$$

(since $p(1-p) \leq \frac{1}{4}$ for $p \in [0,1]$)



$$\Rightarrow \quad \mathbb{P}\left\{\hat{p} - \frac{a}{2\sqrt{n}} < p < \hat{p} + \frac{a}{2\sqrt{n}}\right\} > 1 - \frac{1}{a^2}$$

Ex 1:

$n = 100$

$X = 30 \quad \Rightarrow \quad \hat{p} = 0.3$

$$\mathbb{P}\left\{0.3 - \frac{5}{2\sqrt{100}} < p < 0.3 + \frac{5}{2\sqrt{100}}\right\} > \underbrace{1 - \frac{1}{a^2}}_{0.96}$$

$a = 5$

a	$1 - \frac{1}{a^2}$	
2	0.75	(75%)
5		96%
10		99%

$p = 0.3 \pm 0.25$

$$\underline{P}\left\{ p \in \Big(\overbrace{\hat{p} - \frac{a}{2\sqrt{n}},\ p + \frac{a}{2\sqrt{n}}}^{\text{confidence interval}} \Big) \right\} > 1 - \frac{1}{a^2}$$

estimate of p

half-width

confidence level

full width $= \dfrac{a}{\sqrt{n}}$

$\underline{Q}$: For the same confidence level

If $n' = 4n$

$\Rightarrow \sqrt{n'} = 2\sqrt{n}$

$\Longrightarrow$ reduce conf int width by $\frac{1}{2}$