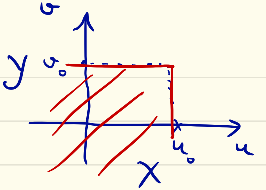
ECE 313: Lecture 30
Joint pdfs of independent random variables (Ch 4.4)

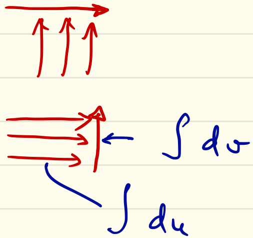


In general — X & Y r.v.

Joint pdf $\quad f_{X,Y}(u,v)$

CDF $\quad F_{X,Y}(u_o, v_o) = P\{X \le u_o,\ Y \le v_o\}$

$$= \int_{-\infty}^{u_o} \int_{-\infty}^{v_o} f_{X,Y}(u,v)\, dv\, du$$



$$= \int_{-\infty}^{v_o} \int_{-\infty}^{u_o} f_{X,Y}(u,v)\, du\, dv$$

Special case: X & Y are independent r.v.

$$F_{X,Y}(u_o,v_o) = P(\underbrace{X \le u_o}_{A},\ \underbrace{Y \le v_o}_{B}) = P(X \le u_o)\, P(Y \le v_o) = \underbrace{F_X(u_o)}_{\text{marginal CDF}}\, \underbrace{F_Y(v_o)}$$

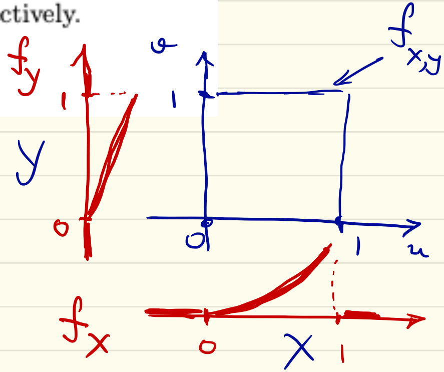
$$f_{X,Y}(u,v) \overset{\text{independent}}{=} \underbrace{f_X(u)}\ \underbrace{f_Y(v)}$$

marginal pdf

$\boxed{(u^n)' = n u^{n-1}}$

5. **[6+10 points]** Suppose X and Y are independent random variables with probability density functions $f_X(u) = \begin{cases} 3u^2 & 0 \le u \le 1 \\ 0 & \text{else,} \end{cases}$ and $f_Y(v) = \begin{cases} 2v & 0 \le v \le 1 \\ 0 & \text{else,} \end{cases}$ respectively.

(a) Find the joint pdf of (X, Y).

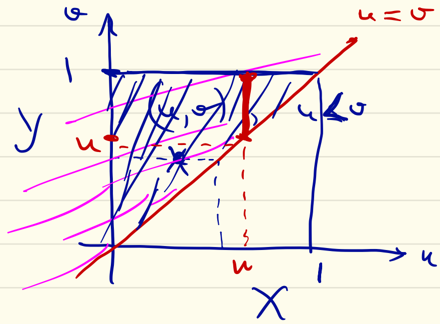


$$f_{X,Y}(u,v) = f_X(u)\, f_Y(v)$$

$$= \begin{cases} 3u^2 \cdot 2v & 0 \le u \le 1,\ 0 \le v \le 1 \\ 0 & \text{else} \end{cases}$$

(b) Find $P\{\underbrace{X < Y}_{A}\}$.

$u \quad v$



$$P(A) = \iint_A f_{X,Y}(u,v)\, du\, dv$$

$$= \int_0^1 \left(\int_u^1 6u^2 v \, dv \right) du$$

$$= \int_0^1 \left(3u^2 v^2\right)\Big|_u^1 \, du = \int_0^1 3u^2(1-u^2)\, du$$

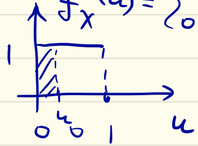
$$= \left(u^3 - \frac{3}{5}u^5\right)\Big|_0^1 = \frac{2}{5}$$

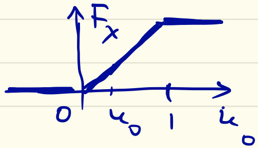
3. **[22 points]** Suppose X and Y are independent random variables such that X is uniformly distributed over the interval $[0,1]$ and Y is exponentially distributed with parameter $\lambda > 0$.

(a) Find the joint CDF $F_{X,Y}$ for all (u,v).

$$F_X(u_0) = \begin{cases} 0 & u_0 < 0 \\ u_0 & 0 \le u_0 \le 1 \\ 1 & u_0 > 1 \end{cases}$$

$$f_X(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{else} \end{cases}$$



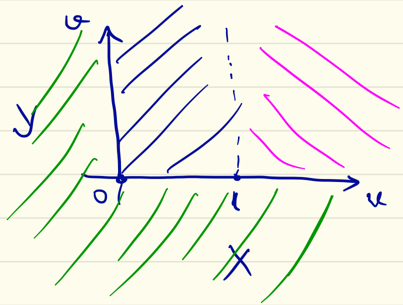


$$F_{X,Y}(u_0, v_0) = \begin{cases} u_0(1-e^{-\lambda v_0}) & 0 \le u_0 \le 1, \quad v_0 \ge 0 \\ 1 \cdot (1-e^{-\lambda v_0}) & u_0 \ge 1, \quad v_0 \ge 0 \\ 0 & u_0 < 0 \text{ or } v_0 < 0 \end{cases}$$

$$F_Y(v_0) = 1 - P(Y \ge v_0) = 1 - e^{-\lambda v_0}$$

$$f_Y(v) = \begin{cases} \lambda e^{-\lambda v} & v \ge 0 \\ 0 & v < 0 \end{cases}$$

(b) Find $P(Y = X)$.



(c) Find $P(Y \le 4X)$.