ECE 313: Lecture 31
Distribution of sums of random variables (Ch 4.5)

Ex: X & Y are r.v. of numbers showing on 2 dice

$$p_X(k) = \frac{1}{6} \qquad 1 \le k \le 6$$

$$p_Y(l) = \frac{1}{6} \qquad \text{"}$$

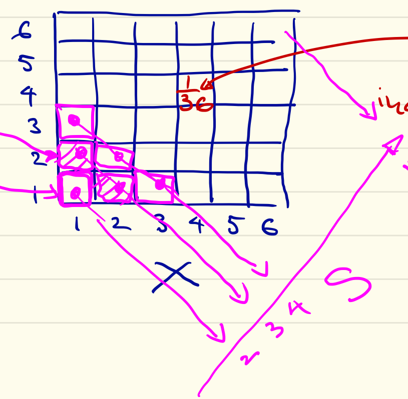
X & Y are independent

Consider: $S = X + Y$. Question: distribution (pmf) of S?

S: $2 = 1+1$

$3 = 1+2 = 2+1$

$4 =$



$$p_{X,Y}(k,l) \overset{\text{independent}}{=} p_X(k)\, p_Y(l)$$

$$p_S(m) = \sum_{k+l=m} p_{X,Y}(k,l)$$

$$= \sum_k p_{X,Y}(k, \underbrace{m-k}_{l})$$

Furthermore, if X & Y are independent:

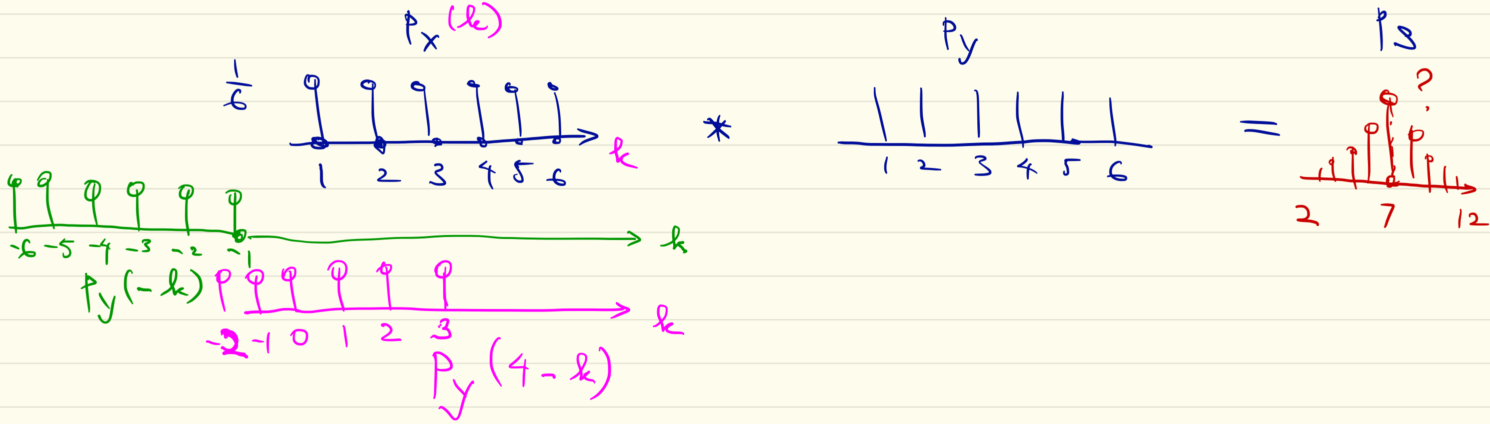
$$p_S(m) = \sum_k p_X(k)\, p_Y(m-k) \overset{\text{def}}{=} (p_X * p_Y)(m)$$

Ah, Convolution!

X & Y indep.

$$\boxed{S = X + Y}$$

Pictorial view of convolution



Ex:

$$X \sim \text{Poisson}(\lambda_1) \qquad p_X(k) = \frac{\lambda_1^k e^{-\lambda_1}}{k!}$$

$$Y \sim \text{Poisson}(\lambda_2) \qquad p_Y(\ell) = \frac{\lambda_2^\ell e^{-\lambda_2}}{\ell!}$$

X & Y independent $\qquad (k, \ell \geq 0)$

$S = X + Y \sim$?

$$p_S(m) = (p_X * p_Y)(m)$$

$$= \sum_{k=0}^{m} p_X(k)\, p_Y(m-k)$$

$$= \sum_{k=0}^{m} \frac{\lambda_1^k e^{-\lambda_1}}{k!} \frac{\lambda_2^{m-k} e^{-\lambda_2}}{(m-k)!}$$

$\lambda = \lambda_1 + \lambda_2$

$$= e^{-\underbrace{(\lambda_1+\lambda_2)}_{\lambda}} \frac{\lambda^{m}}{m!} \left[\underbrace{\sum_{k=0}^{m} \frac{m! \left(\frac{\lambda_1}{\lambda}\right)^k \left(\frac{\lambda_2}{\lambda}\right)^{m-k}}{k!\,(m-k)!}}_{\sum_{k=0}^{m} \text{Binom}\left(m, \frac{\lambda_1}{\lambda}\right)[k] \;=\; 1} \right]$$

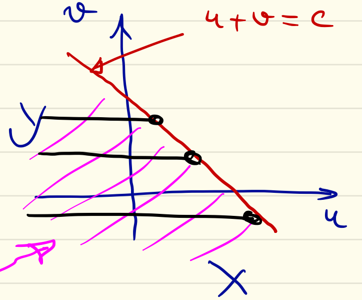
$\Rightarrow \quad S \sim \text{Poisson}(\lambda_1 + \lambda_2)$

(*) X & Y are continuous-type r.v.

joint pdf $f_{X,Y}(u,v)$

$S = X + Y$ has pdf

$$f_S(c) = ?$$



$$F_S(c) = P(S \le c)$$

$$= P(\underbrace{X + Y \le c})$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{c-v} f_{X,Y}(u,v)\, du\, dv$$

$$f_S(c) = \frac{dF_S(c)}{dc} \qquad \frac{d}{dc}$$

$$= \int_{-\infty}^{+\infty} f_{X,Y}(c-v, v)\, \cancel{du}\, dv$$

if X & Y ind. $=$

$$\int_{-\infty}^{+\infty} f_X(c-v)\, f_Y(v)\, dv$$

Convolution $(f_X * f_Y)(c)$