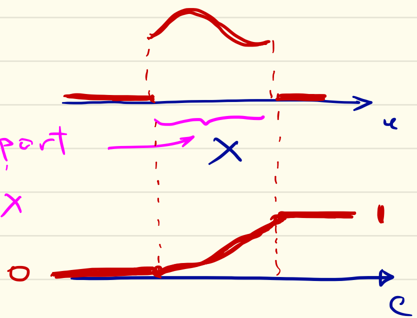
ECE 313: Lecture 33
Review #2: Key concepts from Continuous & Joint Distn

* pdf f_X & CDF F_X

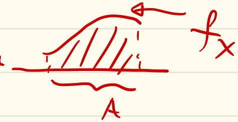
$$f_X(u) = \frac{d F_X(u)}{du}$$



Support of f_X

$$F_X(c) = \int_{-\infty}^{c} f_X(u)\,du$$
$$= P\{X \le c\}$$

* $P\{X \in A\} = \int_A f_X(u)\,du$ = area



$$P\{\underbrace{(X,Y) \in A}\} = \iint_A f_{X,Y}(u,v)\,du\,dv$$

Ex: $X \le Y$

$3X + 4Y > 7$

$X = u_0$

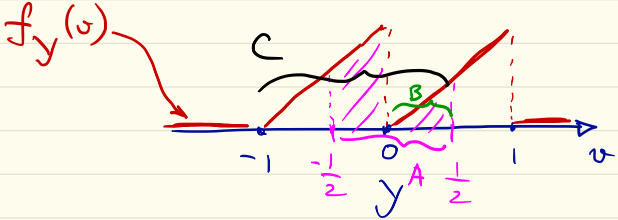
* $$E[g(X)] = \int_{-\infty}^{+\infty} g(u)\, f_X(u)\,du$$

(b) **[24 points]** $\mathcal{Y}$ denotes a random variable with probability density function

$$f_{\mathcal{Y}}(v) = \begin{cases} 1+v, & -1 \le v \le 0, \\ v, & 0 < v \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $P\left\{\underbrace{|\mathcal{Y}| < \frac{1}{2}}_{A}\right\}$, $P\left\{\mathcal{Y} > 0 \,\middle|\, \mathcal{Y} < \frac{1}{2}\right\}$, and $\mathsf{E}[\mathcal{Y}]$.

$$P\left\{|\mathcal{Y}| < \frac{1}{2}\right\} = \frac{1}{2}$$

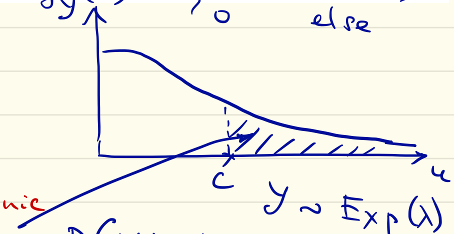


$$P\left\{\mathcal{Y} > 0 \,\middle|\, \mathcal{Y} < \frac{1}{2}\right\} = \frac{P\{\overbrace{y>0,\ y<\frac{1}{2}}^{B}\}}{P\{\underbrace{y<\frac{1}{2}}_{C}\}} = \frac{\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}}{\frac{1}{2}\cdot 1\cdot 1 + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}} = \frac{1}{5}$$

$$E[y] = \int y\, f_y(y)\, dy = \int_{-1}^{0} y(1+y)\,dy + \int_{0}^{1} y\cdot y\, dy$$

5. Let $X \sim U[0,1]$ and Y is an exponential RV with parameter λ. Assume that $Y = g(X)$. Determine $g(\cdot)$.

$$f_Y(u) = \begin{cases} \lambda e^{-\lambda u} & u \geq 0 \\ 0 & \text{else} \end{cases}$$



$Y \sim Exp(\lambda)$

$P(Y \geq c) = e^{-\lambda c}$

Steps for func. of r.v.

0. X : pdf, CDF

1. $Y = g(X)$ | assume g is monotonic

$$\Rightarrow F_Y(c) = P(Y \leq c) = P(X \leq g^{-1}(c))$$

$$= F_X(g^{-1}(c))$$

2. $f_Y(u) = F_Y'(u)$

In this prob, we know

$$Y \sim Exp(\lambda) \rightarrow F_Y(c) = 1 - e^{-\lambda c}$$

$$F_X(g^{-1}(c)) = g^{-1}(c)$$

$$X \sim Uni[0,1] \rightarrow F_X(c) = c \quad (0 \leq c \leq 1)$$

$c = g(x)$ $\quad x = g^{-1}(c)$ Hence $\quad g^{-1}(c) = 1 - e^{-\lambda c} \iff 1 - e^{-\lambda c} = x$, $\quad c = -\ln(1-x)/\lambda = g(x)$

Consider a Poisson process with arrival rate 2 per second. Let A denote the event that there is exactly *one arrival* in the time interval (0, T] and B the event that there are *no arrivals* in the time interval (0.5T, 1.5T].

(a) What are the values of P(A) and P(B)?

(b) Find P(B | A).

$X \sim \mathrm{Poi}(\lambda)$; $p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}$

$\sim \mathrm{Bern}(\lambda h) = \begin{cases} 1 & \text{w.p. } \lambda h \\ 0 & \text{else} \end{cases}$

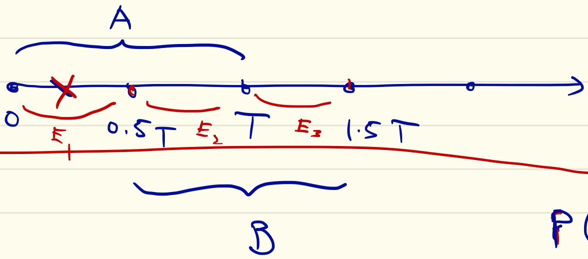
t

\# arrivals $\sim \mathrm{Poi}\left(\overbrace{\frac{t}{h}\cdot\lambda h}^{\lambda t}\right)$

N_i = # arrivals in E_i

$\sim \mathrm{Exp}(\lambda)$

$P(N_1 = 1)\cdot P(N_2 = 0)\cdot P(N_3 = 0)$

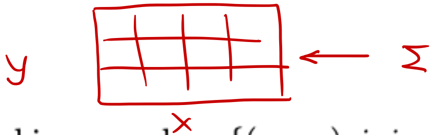


$P(A) = \mathrm{Poi}(\lambda T)[1]$

$= e^{-\lambda T}\cdot\frac{(\lambda T)^1}{1!}$

$P(B \mid A) = \frac{P(AB)}{P(A)} = e^{-\lambda T/2}\cdot\frac{(\lambda T/2)^1}{1!} \times \ldots$

(g) Let (X, Y) be a discrete random vector taking on values $\{(u_i, v_j), i, j = 0, 1, \ldots\}$. Assume that the joint pmf of X and Y is $p_{X,Y}(u_i, v_j)$ and the pmf of X is $p_X(u_i)$. Then $p_X(u_i) \geq p_{X,Y}(u_i, v_j)$ for all v_j. **T/F**

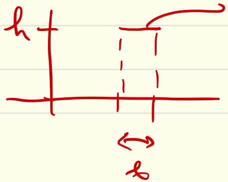


$p_{X|Y}$

(h) Let (X, Y) be a discrete random vector taking on values $\{(u_i, v_j), i, j = 0, 1, \ldots\}$. Assume that the joint pmf of X and Y is $p_{X,Y}(u_i, v_j)$ and the conditional pmf of X given that $Y = v_j$ is $p_{X|Y}(u|v_j)$. Then $p_{X|Y}(u|v_j) \geq p_{X,Y}(u, v_j)$ for all values of u and v_j. **T/F**

(i) Let (X, Y) be a continuous random vector defined over the entire 2D plane. Assume that the joint pdf of X and Y is $f_{X,Y}(u, v)$ and the pdf of X is $f_X(u)$. Then $f_X(u) \geq f_{X,Y}(u, v)$ for all values of v. **T/F**

(j) Let (X, Y) be a continuous random vector defined over the entire 2D plane. Assume that the joint pdf of X and Y is $f_{X,Y}(u, v)$ and the conditional pdf of X given that $Y = v_0$ with $f_Y(v_0) > 0$ is $f_{X|Y}(u|v_0)$. Then $f_{X|Y}(u|v_0) \geq f_{X,Y}(u, v_0)$ for all values of u and v_0. **T/F**



$\int_A f = h \cdot b \gtrless h$