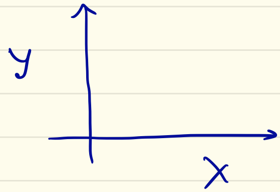
ECE 313: Lecture 35

Correlation and covariance: scaling properties and covariances of sums (Ch 4.8)

X, Y joint random variables

So far, need $f_{X,Y}(u, v)$



Ex: X: house size (in ft^2)
Y: house price (in \$)

$\Rightarrow$ Motivation: a number that quantifies the statistical relation between X & Y

Def: Covariance between X & Y:

$$\underline{\mathrm{Cov}(X, Y)} = E[(X - \underbrace{E[X]}_{\mu_X})(Y - \underbrace{E[Y]}_{\mu_Y})]$$

Properties

① $$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2 - 2\underbrace{E[X]}_{\mu_X}.X + \underbrace{E[X]^2}_{\mu_X^2}]$$

$\parallel$

$\mathrm{Cov}(X, X)$

$$= E[X^2] - 2E[X].E[X] + E[X]^2$$

$$= E[X^2] - (E[X])^2$$

(*) Properties (cont.)

(2) $Cov(X,Y) = E[XY - E[X]Y - E[Y].X + E[X]E[Y]]$

$= E[XY] - E[X]E[Y] - E[Y]E[X] + E[X]E[Y]$

$= E[XY] - E[X]E[Y]$

(3) Cov is linear with respect to each arguments:

$$Cov(\underbrace{aX + bY + c}_{Z}, Y) = a\,Cov(X,Y) + b\,Cov(Y,Y) + c\,\underbrace{Cov(1,Y)}_{=0}$$

Proof: $Cov(Z,Y) = E[(Z - E[Z])(Y - E[Y])]$

We have: $E[Z] = a\,E[X] + b\,E[Y] + c$

$\Rightarrow Z - E[Z] = a(X - E[X]) + b(Y - E[Y]) + c.0$

$\Rightarrow E[(Z-E[Z])(Y-E[Y])] = a\,E[(X-E[X])(Y-E[Y])]$

$+ b\,E[(Y-E[Y])(Y-E[Y])]$

④ $\mathrm{Cov}(aX + bY, cZ + dW)$

$= \mathrm{Cov}(aX, cZ + dW) + \mathrm{Cov}(bY, cZ + dW)$

$= \mathrm{Cov}(aX, cZ) + \mathrm{Cov}(aX, dW) + \mathrm{Cov}(bY, cZ) + \mathrm{Cov}(bY, dW)$

$= ac\,\mathrm{Cov}(X, Z) + ad\,\mathrm{Cov}(X, W) + bc\,\mathrm{Cov}(Y, Z) + bd\,\mathrm{Cov}(Y, W)$

$\mathrm{Cov}(X, Y)$ can be viewed as the dot product $\langle X, Y \rangle$

⊛ Def: Correlation coefficient between X & Y

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\ \mathrm{Var}(Y)}} = \frac{[ft^2 \cdot \$]}{\sqrt{(ft^2)^2\ (\$)^2}}$$

4.17. **[Deducing a covariance from variances]**
Consider random variables X and Y on the same probability space.

(a) If $\mathrm{Var}(X+2Y)=40$ and $\mathrm{Var}(X-2Y)=20$, what is $\mathrm{Cov}(X,Y)$?

(b) In part (a), determine $\rho_{X,Y}$ if $\mathrm{Var}(X)=2\cdot\mathrm{Var}(Y)$.

a)

$$\begin{cases} \mathrm{Var}(X+2Y) = \mathrm{Cov}(X+2Y, X+2Y) \\ \quad = \mathrm{Cov}(X,X) + 4\,\mathrm{Cov}(X,Y) + 4\,\mathrm{Cov}(Y,Y) \\ \mathrm{Var}(X-2Y) = \mathrm{Cov}(X,X) - 4\,\mathrm{Cov}(X,Y) + 4\,\mathrm{Cov}(Y,Y) \end{cases}$$

$$(-) \Rightarrow \quad \mathrm{Cov}(X,Y) = \frac{\mathrm{Var}(X+2Y) - \mathrm{Var}(X-2Y)}{8} = \frac{40-20}{8}$$

b) $$(+) \Rightarrow \quad \mathrm{Var}(X) + 4\,\mathrm{Var}(Y) = \frac{\mathrm{Var}(X+2Y) + \mathrm{Var}(X-2Y)}{2}$$

$$= \frac{40+20}{2}$$

4.19. **[Working with covariances]**
Suppose X, Y, and Z are random variables, each with mean zero and variance 20, such that $\text{Cov}(X, Y) = \text{Cov}(X, Z) = 10$ and $\text{Cov}(Y, Z) = 5$. Be sure to show your work, as usual, for all parts below.

$E[X] = E[Y] = E[Z] = 0$

$Var(X) = Var(Y) = Var(Z) = 20$

(a) Find $\text{Cov}(X + Y, X - Y)$.

(b) Find Cov(3X+Z,3X+Y). $= 9\,Cov(X, X) + 3\,Cov(X, Y) + 3\,Cov(X, Z) + Cov(Y, Z)$

(c) Find $E[(X + Y)^2]$.

Note that:

(c) $\underbrace{Var(X+Y)} = E[(X+Y)^2] - (\underbrace{E[X+Y]}_{(E[X] + E[Y])^2})^2$

$= Cov(X+Y, X+Y)$

$= \underbrace{Cov(X, X)}_{Var(X)} + 2\,Cov(X, Y) + \underbrace{Cov(Y, Y)}_{Var(Y)}$