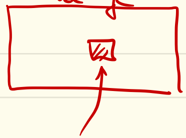
ECE 313: Lecture 38: Application of MMSE linear estimator

Application :

$$\underset{\text{observed noisy signal}}{\underset{\sim}{X}} = \underset{\text{clean, unobserved signal}}{\underset{\sim}{Y}} + \underset{\sim}{N} \quad \text{noise} \sim \mathcal{N}(0, \sigma_N^2)$$



Goal : Given $X \longrightarrow \hat{Y}$ (Signal Denoising)

Setting : Restrict to linear estimator

$$\hat{Y} = L(X) = a X + b$$

constants

$$\text{MSE} = E_{Y,N}\left[(\hat{Y} - Y)^2 \right]$$

MMSE linear predictor $\hat{Y} = \mu_Y + \dfrac{\text{Cov}(X,Y)}{\text{Var}(X)} (X - \mu_X)$

$\Rightarrow$ Need μ_X, μ_Y, $\text{Cov}(X,Y)$, $\text{Var}(X)$

We always have $\mu_X = \mu_Y + \mu_N$

In practice $\hat{\mu}_X$ and $\hat{\sigma}_X^2$ can be estimated from data

Assume Y & N are uncorrelated

$$\text{Cov}(X,Y) = \text{Cov}(Y+N, Y)$$
$$= \text{Cov}(Y,Y) + \cancel{\text{Cov}(N,Y)}^{\,0}$$
$$= \sigma_Y^2$$

$$X = Y + N \qquad Y \text{ \& } N \text{ are uncorrelated}$$

$$\begin{aligned}
Var(X) &= Cov(X, X) \\
&= Cov(Y+N, Y+N) \\
&= Cov(Y, Y) + 2\, \cancel{Cov(Y, N)}^{\,0} + Cov(N, N) \\
&= \sigma_Y^2 + \sigma_N^2
\end{aligned}$$

MMSE linear estimator $\quad X \longrightarrow \hat{Y}$

$$\hat{Y} = \mu_Y + \frac{\sigma_Y^2}{\sigma_Y^2 + \sigma_N^2} (X - \mu_X)$$

$$\Leftrightarrow \quad (\hat{Y} - \mu_Y) = \frac{\sigma_Y^2}{\sigma_Y^2 + \sigma_N^2} (X - \mu_X)$$

Note that:

Signal-to-Noise ratio $\quad SNR = \dfrac{\sigma_Y^2}{\sigma_N^2}$.

If $SNR \rightarrow$ big
$\hat{Y} - \mu_Y = X - \mu_X$ (i.e. trust X)

If $SNR \rightarrow 0$
$\hat{Y} - \mu_Y = 0$ (I.e. ignore X)

ECE 313: Lecture 38: Law of large numbers (Ch 4.10.1)

Law of Large Number (LLN)

(IID) i.i.d

$X_1, X_2, \dots, X_n$ are independent & identically distributed

$$S = X_1 + X_2 + \dots + X_n \quad \text{or} \quad \frac{S}{n}$$

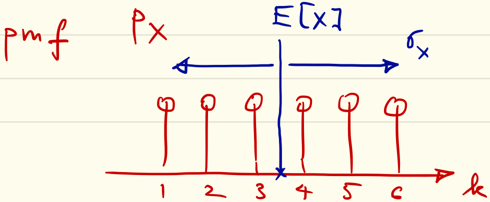
Ex: Throw a dice: $X_i \in \{1,2,3,4,5,6\}$

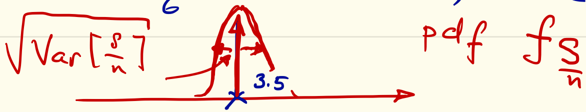
$k = 1, 2, \dots, 6$: $P(X_i = k) = \frac{1}{6}$

$$E[X_i] = \frac{1}{6}(1 + 2 + \dots + 6) = 3.5$$

$$Var[X_i] = E[X_i^2] - (E[X_i])^2$$

$$= \frac{1}{6}(1^2 + 2^2 + \dots + 6^2) - (3.5)^2 = 2.9$$





$$E[S] = E[X_1 + X_2 + \dots + X_n] = E[X_1] + \dots + E[X_n] = n\,E[X_i]$$

$$\Rightarrow \quad E\left[\frac{S}{n}\right] = E[X_i]$$

$$\begin{aligned}
\mathrm{Var}[S] &= \mathrm{Var}[X_1 + \dots + X_n] \\
&= \mathrm{Cov}\left(\sum_{i=1}^{n} X_i \, , \, \sum_{i=1}^{n} X_i\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} \mathrm{Var}(X_i) \quad = \quad n \, \mathrm{Var}(X_i)
\end{aligned}$$

$$\Longrightarrow \quad \mathrm{Var}\left[\frac{S}{n}\right] = \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} \, \mathrm{Var}(X_i)$$

Chebyches: $P\left(\left| \frac{S}{n} - E[X_i] \right| < \delta \right) \leq \dfrac{\mathrm{Var}\left(\frac{S}{n}\right)}{\delta^2} = \dfrac{\mathrm{Var}(X_i)}{n\delta^2}$