ECE 313: Lecture 39
Central limit theorem (Ch 4.10.2)

X_i can have ANY pdf

$$\begin{cases} E[X_i] = \mu \\ Var[X_i] = \sigma^2 \end{cases}$$

n i.i.d. $X_1, X_2, \dots, X_n$

Consider:

(sum) $S_n = X_1 + X_2 + \dots + X_n$

(average/mean) $\frac{S_n}{n}$

$$E[S_n] = \sum_{i=1}^{n} E[X_i] = n\mu$$

$$Var[S_n] = \sum_{i=1}^{n} Var[X_i] = n\sigma^2$$

$$E\left[\frac{S_n}{n}\right] = \mu$$

$$Var\left[\frac{S_n}{n}\right] = \frac{\sigma^2}{n}$$

LLN:

$$P\left(\left|\frac{S_n}{n} - \underset{E[\frac{S_n}{n}]}{\mu}\right| > t\right) \le \underset{Var[\frac{S_n}{n}]}{\frac{\sigma^2}{n}} \cdot \frac{1}{t^2}$$

$$P\left(\frac{S_n}{n} \in I_t\right) \to 1$$

$n \to$ large

pdf of $\frac{S_n}{n}$

I.e. $\frac{S_n}{n}$ piles up on interval I_t around μ for any $t > 0$

Central limit theorem:

Normal (Gaussian) dist.

$$\frac{S_n}{n} \xrightarrow{n \to \infty} \mathcal{N}\Big(\underbrace{E\Big[\frac{S_n}{n}\Big]}_{\mu} ; \underbrace{Var\Big[\frac{S_n}{n}\Big]}_{\frac{\sigma^2}{n}} \Big)$$

CLT (Gaussian approximation)

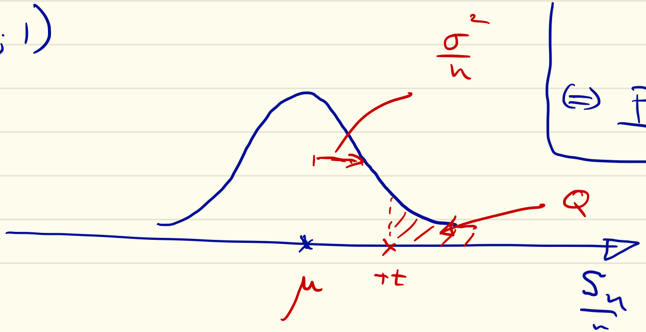
$$P\Big(\frac{S_n}{n} - \mu > t \Big) \approx P\Big(Z > \underbrace{\frac{t}{\sqrt{\frac{\sigma^2}{n}}}}_{= c} \Big) = Q(c)$$

$$\frac{S_n}{n} \to N\Big(\mu ; \frac{\sigma^2}{n}\Big)$$

$$Z = \frac{\frac{S_n}{n} - \mu}{\sqrt{\frac{\sigma^2}{n}}} \approx N(0;1)$$

$$\boxed{P\Big(\frac{S_n}{n} - \mu > \underbrace{\frac{c\,\sigma}{\sqrt{n}}}_{t} \Big) = Q(c)}$$

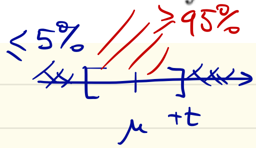
$$\Leftrightarrow P\Big(\frac{S_n - n\mu}{\sqrt{n}\,\sigma} > c \Big) = Q(c)$$



9. **[8+8 points]** Suppose $X_1, X_2, ..., X_n$ are independent, identically distributed random variables, each with mean $\mu = 2$ and variance $\sigma^2 = 4$. Let $S_n = X_1 + X_2 + ... + X_n$. Determine a condition on n so the probability the sample average $\frac{S_n}{n}$ is within 1% of the mean ($\mu = 2$), is greater than 0.95.

(a) Solve the problem using the form of the law of large numbers based on the Chebychev inequality.

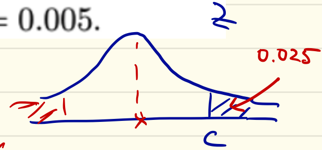
Use $$P\left(\left|\frac{S_n}{n}-\mu\right| > \underbrace{t}_{0.01\mu}\right) \leq \frac{\sigma^2}{nt^2}$$



We want $$\frac{\sigma^2}{n \times (0.01\mu)^2} \leq 0.05 \iff n \geq \frac{\sigma^2}{0.05 \times (0.01)^2 \mu^2}$$

$$\iff n \geq \frac{4}{5 \cdot 10^{-2} \cdot 4 \cdot 10^{-4}} = 2 \cdot 10^5$$

(b) Solve the problem using the Gaussian approximation for S_n according to the central limit theorem. The following Q-function table is provided in case needed: $Q(1.28) = 0.1, Q(1.65) = 0.05, Q(1.96) = 0.025, Q(2.33) = 0.01, Q(2.58) = 0.005$.



$$P\left(\frac{S_n - n\mu}{\sqrt{n\sigma^2}} > c\right) \approx \underbrace{Q(c)}$$ $$= 0.025 \quad \left(= \frac{1-0.95}{2}\right)$$

We want

$\Rightarrow \boxed{c = 1.96}$ $$P(S_n - n\mu > 0.01\, n\mu) > 0.025$$

$$\Rightarrow c \cdot \sqrt{n\sigma^2} = 0.01\, n\mu$$

(d) Suppose $U_1, U_2, \ldots U_n$ is a sequence of i.i.d. random variables such that each U_k has a uniform distribution over $[0, c]$. Consider the product $\prod_{k=1}^{n} U_k$ as $n \to \infty$.

$0 \quad \frac{c}{2} \quad c$

TRUE	FALSE		
☐	☐	If $c = 2$, $P(\prod_{k=1}^{n} U_k > \delta) \to 0$ as $n \to \infty$ for any $\delta > 0$.	True
☐	☐	If $c = 3$, $P(\prod_{k=1}^{n} U_k > \delta) \to 0$ as $n \to \infty$ for any $\delta > 0$.	False

$\prod_{k=1}^{n} U_k = P_n$

Trick :

$$\ln P_n = \underbrace{\ln U_1}_{X_i} + \ln U_2 + \cdots + \ln U_n$$

$E[X_i]$

$\Rightarrow$ (LOTUS) $\displaystyle\int_0^c \ln(u) \frac{1}{c} \, du$

$$= \frac{1}{c} \left[u \ln(u) - u \right] \Big|_0^c$$

$$= \ln(c) - 1$$

We want to know:

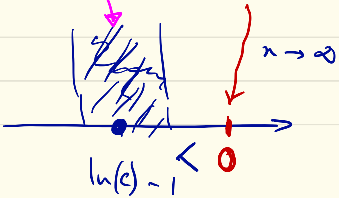
$$P(\ln P_n > \ln \delta)$$

$$= P\left(\frac{\ln P_n}{n} > \frac{\ln \delta}{n} \right)$$

$n \to \infty$

... Use LLN for $\frac{\ln P_n}{n}$

If $c = 2$



If $c = 3$