ECE 313: Lecture 25
The distribution of a function of a random variable (Ch 3.8.1) [Cont.]
Generating random variables with a specified distribution (Ch 3.8.2)
Applications: quantization, histogram equalization/shaping

Review prob:

$f_Y = \mathcal{N}(\mu; \sigma^2)$

$$f_X(u) = \begin{cases} K \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right) & a \le u \le b \\ 0 & \text{else} \end{cases}$$

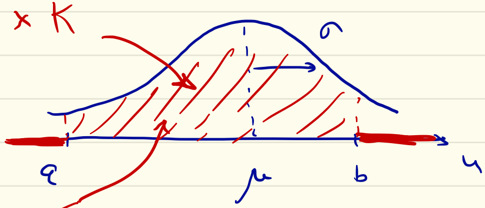
truncated

Find a, b, K

Need

$$K = \frac{1}{A} \quad \text{so that}$$

$$\int_a^b f_X(u)\, du = 1$$



A = Area of Gaussian from $[a, b]$

$$A = F_Y(b) - F_Y(a)$$

$$= \left[1 - Q\left(\frac{b-\mu}{\sigma}\right)\right] - \left[1 - Q\left(\frac{a-\mu}{\sigma}\right)\right]$$

$$= Q\left(\frac{a-\mu}{\sigma}\right) - Q\left(\frac{b-\mu}{\sigma}\right)$$

lookup table

Given X $\longrightarrow$ $Y = g(X)$ $\quad Y$

pdf $\quad f_X(u) = F_X'(x)$

CDF $\quad F_X(c) = \int_{-\infty}^{c} f_X(x)\,dx$

? $\left\{ \begin{array}{ll} \text{pdf} & f_Y \\ \text{CDF} & F_Y \end{array} \right.$

$E[Y]$, $Var[Y]$

Case 1 : Y continuous - type

(CDF) ① $F_Y(c) \overset{def}{=} P\{Y \leq c\}$

$= P\{g(X) \leq c\}$

(pdf) ② $f_Y(c) = \dfrac{dF_Y(c)}{dc}$ $= \int_a^b \ldots dx$ (using f_X or F_X)

Case 2 : Y discrete - type

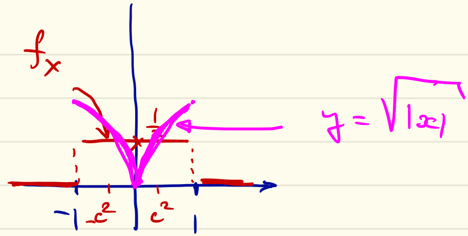
(pmf) ① $P\{Y = k\} = P\{g(X) = k\}$

$= \int_a^b \ldots \quad f_X$

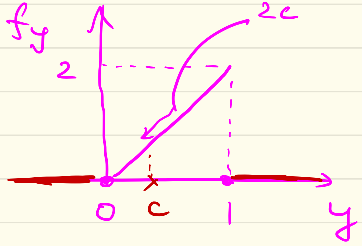
$E[Y]$ (LOTUS) $\quad E[g(X)] = \int_{-\infty}^{+\infty} g(u)\, f_X(u)\, du$

Ex: $X \sim \text{Uniform } [-1, 1]$

$$f_X(u) = \begin{cases} \frac{1}{2} & -1 \le u \le 1 \\ 0 & \text{else} \end{cases}$$



For $Y = \sqrt{|X|}$, find F_Y, f_Y, $E[Y]$



For $0 \le c \le 1$:

$$F_Y(c) \overset{\text{def}}{=} P\{Y \le c\}$$

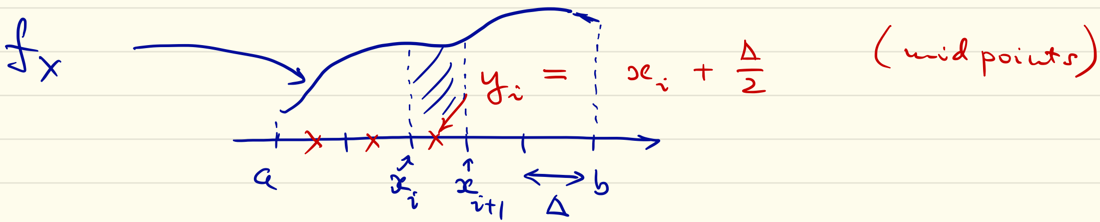
$$\overset{\text{sub}}{=} P\{\sqrt{|X|} \le c\}$$

$$= P\{-c^2 \le X \le c^2\}$$

$$= \int_{-c^2}^{c^2} \cancelto{\frac{1}{2}}{f_X(u)}\, du = c^2$$

$$f_Y(c) = \frac{dF_Y(c)}{dc} = 2c$$

Application: (Scalar) uniform quantization



$$X \xrightarrow{\text{quantize}} Y = q(X) = y_i \quad \text{if } x_i \leq x \leq x_{i+1}$$

$$\Delta = \frac{b-a}{n}$$

$$a = x_0 \quad ; \quad x_i = a + i\Delta \quad (0 \leq i \leq n), \quad x_n = b$$

⊛ Y is discrete type

$$P\{Y = y_i\} = P\{x_i \leq X \leq x_{i+1}\} = \int_{x_i}^{x_{i+1}} f_X(u)\,du$$

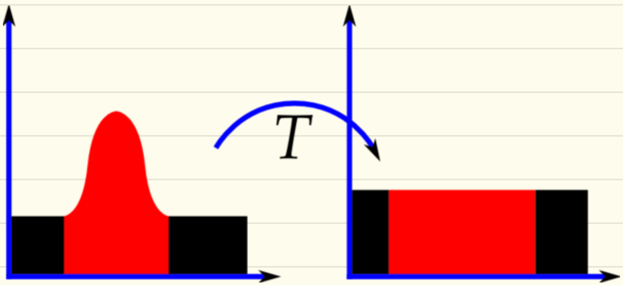
⊛ Quantization error (Mean Square Error)

$$E[(Y-X)^2] \overset{\text{LOTUS}}{=} \int_{-\infty}^{+\infty} (q(x)-x)^2 f_X(x)\,dx =$$

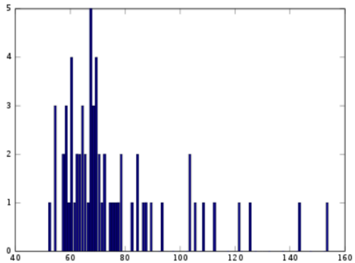
$$= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} (y_i - x)^2 f_X(x)\,dx$$

Application: Histogram Equalization (see: https://en.wikipedia.org/wiki/Histogram_equalization)

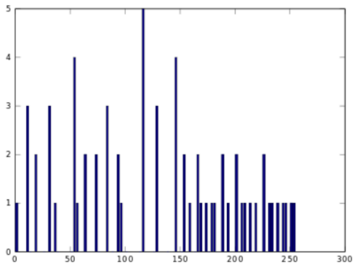
52	55	61	59	79	61	76	61
62	59	55	104	94	85	59	71
63	65	66	113	144	104	63	72
64	70	70	126	154	109	71	69
67	73	68	106	122	88	68	68
68	79	60	70	77	66	58	75
69	85	64	58	55	61	65	83
70	87	69	68	65	73	78	90



0	12	53	32	146	53	174	53
57	32	12	227	219	202	32	154
65	85	93	239	251	227	65	158
73	146	146	247	255	235	154	130
97	166	117	231	243	210	117	117
117	190	36	190	178	93	20	170
130	202	73	20	12	53	85	194
146	206	130	117	85	166	182	215



Histogram of Original image

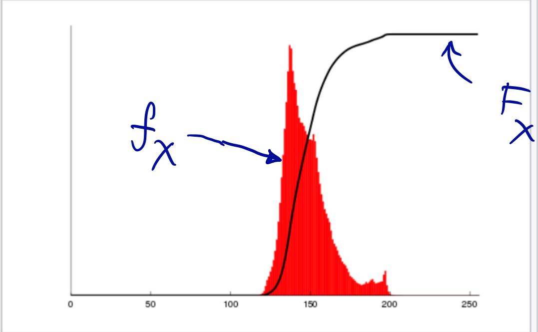


Histogram of Equalized image

X

Image Intensity Transformation

Before Histogram Equalization

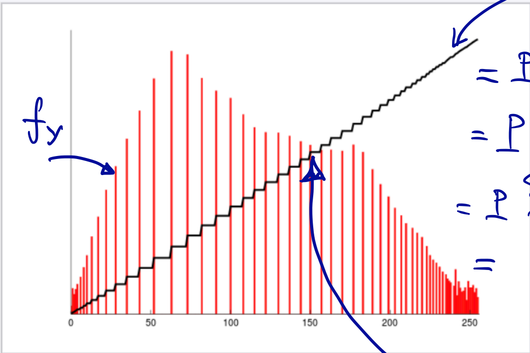


Corresponding histogram (red) and cumulative histogram (black)

$y = g(x)$

(monotonic increasing)

After Histogram Equalization



Corresponding histogram (red) and cumulative histogram (black)

$F_Y(c)$

$= P\{Y \le c\}$

$= P\{g(X) \le c\}$

$= P\{X \le g^{-1}(c)\}$

$= F_X(g^{-1}(c))$

linear CDF

$\Rightarrow$ By choosing $g = F_X$ $\Rightarrow F_Y(c) = F_X(F_X^{-1}(c)) = c$