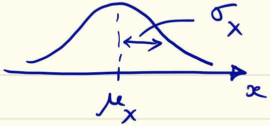
ECE 313: Lecture 40
Joint Gaussian distribution



X is $N(\mu_X ; \sigma_X^2) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{(x-\mu_X)^2}{\sigma_X^2}}$

Standardized $\quad Z = \frac{X-\mu_X}{\sigma_X} \sim N(0;1)$

$\Leftrightarrow X = \mu_X + \sigma_X Z$

$\circledast$ Two R.V. X & Y are jointly Gaussian

Def: ① Via joint pdf $f_{X,Y}(x,y)$

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot \sigma_X \cdot \sigma_Y} \exp\left(- \frac{\left(\frac{x-\mu_X}{\sigma_X^2}\right)^2 + \left(\frac{y-\mu_Y}{\sigma_Y^2}\right)^2 - 2\rho_{X,Y}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)}{2(1-\rho_{X,Y}^2)} \right)$$

② (Equivalent def):

Standard $\begin{cases} Z \sim N(0,1) \\ W \sim N(0,1) \\ Z \text{ \& } W \text{ are indep.} \end{cases}$ $\Rightarrow$ general $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} + \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} Z \\ W \end{pmatrix}$

$\Rightarrow X = \mu_X + aZ + bW$

$Y = \mu_Y + cZ + dW$

$$f_{Z,W}(z,w) = f_Z(z)\, f_W(w) = \frac{1}{2\pi} \exp\left(-\left(z^2 + w^2\right)\right)$$

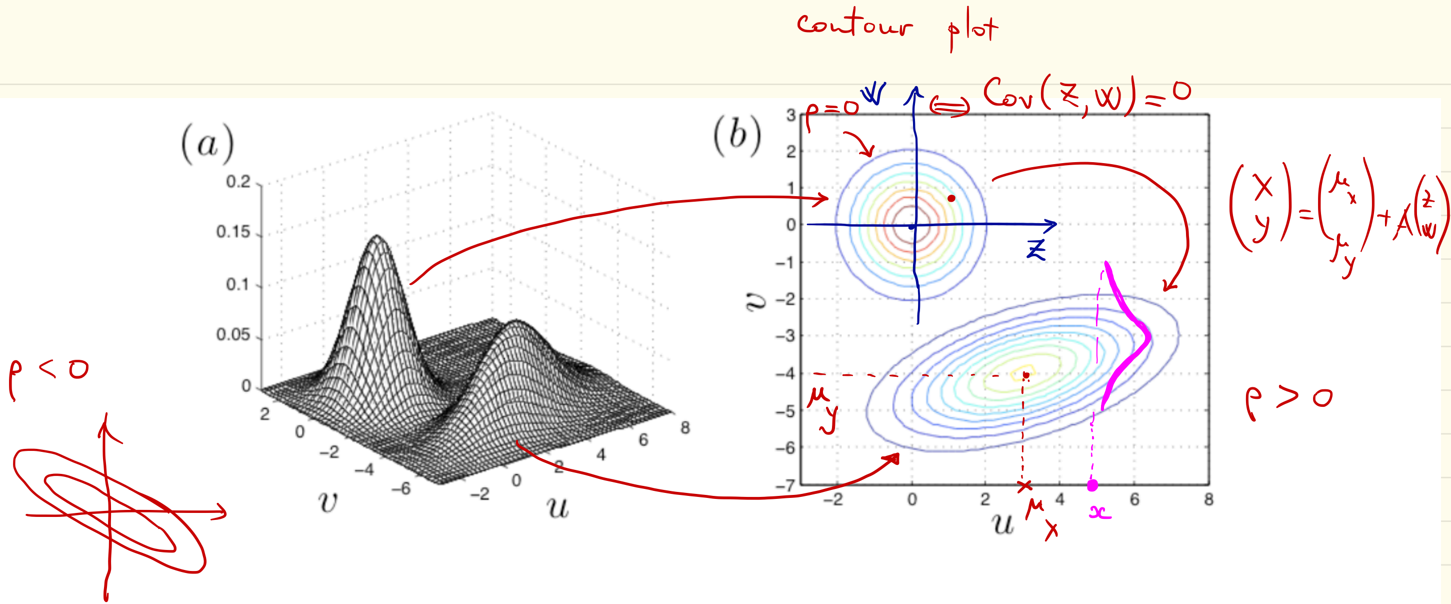


Figure 4.27: (a) Mesh plots of both the standard bivariate normal, and the bivariate normal with $\mu_X = 3, \mu_Y = -4, \sigma_X = 2, \sigma_Y = 1, \rho = 0.5$, shown on the same axes. (b) Contour plots of the same pdfs.

$$f_{X,Y}(x,y) = \frac{1}{2\pi \ldots} \exp\left(- \frac{(x-\mu_X)^2}{\sigma_X \ldots} \right) \longrightarrow f_X(x) = \mathcal{N}(\mu_X ; \sigma_X^2)$$

$$\mathrm{Cov}(X,Y) = \underbrace{\rho_{X,Y}}_{\rho} \cdot \sigma_X \cdot \sigma_Y$$

ρ : correlation coefficient

Properties of bivariate Gaussian normal dist.

$$(X, Y) \sim f_{X,Y}(x,y) = \cdots \qquad \overbrace{\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho_{X,Y}}^{5 \text{ parameters}}$$

① marginal
$$\begin{cases} X \sim \mathcal{N}(\mu_X ; \sigma_X) \\ Y \sim \mathcal{N}(\mu_Y ; \sigma_Y) \end{cases}$$

②
$$\underbrace{aX + bY}_{R} \sim \mathcal{N}(\mu_R ; \sigma_R^2)$$

where $\mu_R = E[R] = E[aX + bY] = a\mu_X + b\mu_Y$

$$\sigma_R^2 = Var(R) = Cov(aX+bY, aX+bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab \underbrace{\rho_{X,Y}\sigma_X\sigma_Y}_{Cov(X,Y)}$$

③ Conditional $f_{Y|X=u} \sim \mathcal{N}(E[Y|X=u], Var[Y|X=u])$

where $E[Y|X=u] = \hat{E}[Y|X=u]$ ← linear MMSE pred.

$Var[Y|X=u]$ = corresp. MSE

7. **[7+16+7 points]** Suppose $X \sim N(1,1)$ and $Y \sim N(1,4)$ are independent Gaussian random variables. Define the random variables $Z = 2X + Y$ and $W = X - Y$.

$\mu_x = 1$, $\sigma_x^2 = 1$ $\quad \mu_y = 1$ $\quad \sigma_y^2 = 4$

(a) Find the unconstrained MMSE estimator of Y given X, and the resulting MSE.

$$= E[Y \mid X] = E[Y] = 1$$

(x & Y ind.)

$$\text{MSE} = \text{Var}(Y)$$

(b) Find the unconstrained MMSE estimator of Z given W, and the resulting MSE.

$$= E[Z \mid W] = \hat{E}[Z \mid W]$$

$$= \boxed{\mu_Z + \frac{\text{Cov}(Z,W)}{\text{Var}(W)}(W - \mu_W)}$$

where $\mu_Z = E[2X + Y] = 2E[X] + E[Y]$

$$\text{Cov}(Z,W) = \text{Cov}(2X+Y, X-Y) = 2\,\text{Cov}(X,X) - \text{Cov}(X,Y) - \text{Cov}(Y,Y)$$
$$= 2\sigma_x^2 - \sigma_y^2$$

Similarly for $\text{Var}(W) = \text{Cov}(X-Y, X-Y) = \dots$; $\text{Var}(Z) = \dots$

$$\text{Resulting MSE} = \boxed{\text{Var}(Z) - \frac{\text{Cov}(Z,W)^2}{\text{Var}(W)}} = \dots$$

(c) If instead $W = X - aY$ for some real a and $E[Z|W] = E[Z]$, find a.

$\iff Z$ & W are independent

$$\iff 0 = \text{Cov}(Z, W) = \text{Cov}(2X+Y, X-aY) = 2\sigma_x^2 - a\sigma_y^2 \Rightarrow a = \frac{1}{2}$$