Lecture 7: Frequency Response

Mark Hasegawa-Johnson

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Outline

What is Signal Processing, Really?

- When we process a signal, usually, we're trying to enhance the meaningful part, and reduce the noise.
- **Spectrum** helps us to understand which part is meaningful, and which part is noise.
- **Convolution** (a.k.a. filtering) is the tool we use to perform the enhancement.
- **Frequency Response** of a filter tells us exactly which frequencies it will enhance, and which it will reduce.

Review: Convolution

- A **convolution** is exactly the same thing as a **weighted local average**. We give it a special name, because we will use it very often. It's defined as:

$$y[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

- We use the symbol $*$ to mean "convolution:"

$$y[n] = g[n]*f[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

Review: Spectrum

The **spectrum** of $x(t)$ is the set of frequencies, and their associated phasors,

$$\text{Spectrum}\,(x(t)) = \{(f_{-N}, a_{-N}), \ldots, (f_0, a_0), \ldots, (f_N, a_N)\}$$

such that

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi f_k t}$$

Review: Fourier Series

One reason the spectrum is useful is that **any** periodic signal can be written as a sum of cosines. Fourier's theorem says that any $x(t)$ that is periodic, i.e.,

$$x(t+T_0) = x(t)$$

can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi k F_0 t}$$

which is a special case of the spectrum for periodic signals: $f_k = kF_0$, and $a_k = X_k$, and

$$F_0 = \frac{1}{T_0}$$

- **Fourier Series Analysis** (finding the spectrum, given the waveform):

$$X_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

- **Fourier Series Synthesis** (finding the waveform, given the spectrum):

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}$$

- **DFT Analysis** (finding the spectrum, given the waveform):

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

- **DFT Synthesis** (finding the waveform, given the spectrum):

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}$$

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Frequency Response

The **frequency response**, $G(\omega)$, of a filter $g[n]$, is its output in response to a pure tone, expressed as a function of the frequency of the tone.

Calculating the Frequency Response

- **Output of the filter:**

$$\begin{aligned} y[n] &= g[n] * x[n] \\ &= \sum_m g[m]x[n-m] \end{aligned}$$

- **in response to a pure tone:**

$$x[n] = e^{j\omega n}$$

Calculating the Frequency Response

Output of the filter in response to a pure tone:

$$\begin{aligned} y[n] &= \sum_m g[m]x[n-m] \\ &= \sum_m g[m]e^{j\omega(n-m)} \\ &= e^{j\omega n}\left(\sum_m g[m]e^{-j\omega m}\right) \end{aligned}$$

Notice that the part inside the parentheses is not a function of n. It is not a function of m, because the m gets summed over. It is only a function of ω. It is called the frequency response of the filter, $G(\omega)$.

Frequency Response

When the input to a filter is a pure tone,

$$x[n] = e^{j\omega n},$$

then its output is the same pure tone, scaled and phase shifted by a complex number called the **frequency response** $G(\omega)$:

$$y[n] = G(\omega)e^{j\omega n}$$

The frequency response is related to the impulse response as

$$G(\omega) = \sum_m g[m]e^{-j\omega m}$$

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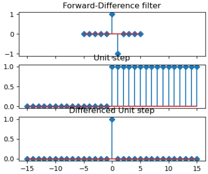
Example: First Difference



Remember that taking the difference between samples can be written as a convolution:

$$y[n] = x[n] - x[n-1] = g[n] * x[n],$$

where

$$g[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example: First Difference

Suppose that the input is a pure tone:

$$x[n] = e^{j\omega n}$$

Then the output will be

$$\begin{aligned} y[n] &= x[n] - x[n-1] \\ &= e^{j\omega n} - e^{j\omega(n-1)} \\ &= G(\omega)e^{j\omega n}, \end{aligned}$$

where

$$G(\omega) = 1 - e^{-j\omega}$$

First Difference Filter at $\omega = 0$

$$G(\omega) = 1 - e^{-j\omega}$$

- Frequency $\omega = 0$ means the input is a constant value:

$$x[n] = e^{j\omega n}|_{\omega=0} = 1$$

- At frequency $\omega = 0$, the frequency response is zero!

$$G(0) = 1 - e^{0} = 0$$

- ...which totally makes sense, because if $x[n] = 1$, then

$$y[n] = x[n] - x[n-1] = 1 - 1 = 0$$

First Difference Filter at $\omega = \pi$

- Frequency $\omega = \pi$ means the input is $(-1)^n$:

$$x[n] = e^{j\pi n} = (-1)^n = \begin{cases} 1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{cases}$$

- At frequency $\omega = \pi$, the frequency response is two!

$$G(\pi) = 1 - e^{j\pi} = 1 - (-1) = 2$$

- ...which totally makes sense, because if $x[n] = (-1)^n$, then

$$y[n] = x[n] - x[n-1] = \begin{cases} 1 - (-1) = 2 & n \text{ is even} \\ (-1) - 1 = -2 & n \text{ is odd} \end{cases}$$

First Difference Filter at $\omega = \frac{\pi}{2}$

Frequency $\omega = \frac{\pi}{2}$ means the input is j^n:

$$x[n] = e^{j\frac{\pi n}{2}} = j^n = \begin{cases} 1 & n \in \{0, 4, 8, 12, \ldots\} \\ j & n \in \{1, 5, 9, 13, \ldots\} \\ -1 & n \in \{2, 6, 10, 14, \ldots\} \\ -j & n \in \{3, 7, 11, 15, \ldots\} \end{cases}$$

The frequency response is:

$$G\left(\frac{\pi}{2}\right) = 1 - e^{j\frac{\pi}{2}} = 1 - j,$$

so $y[n]$ is

$$y[n] = (1-j)e^{j\frac{\pi n}{2}} = (1-j)j^n = \begin{cases} (1-j) & n \in \{0, 4, 8, 12, \ldots\} \\ (j+1) & n \in \{1, 5, 9, 13, \ldots\} \\ (-1+j) & n \in \{2, 6, 10, 14, \ldots\} \\ (-j-1) & n \in \{3, 7, 11, 15, \ldots\} \end{cases}$$

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Superposition and the Frequency Response

The frequency response obeys the principle of **superposition**, meaning that if the input is the sum of two pure tones:

$$x[n] = e^{j\omega_1 n} + e^{j\omega_2 n},$$

then the output is the sum of the same two tones, each scaled by the corresponding frequency response:

$$y[n] = G(\omega_1)e^{j\omega_1 n} + G(\omega_2)e^{j\omega_2 n}$$

Response to a Cosine

There are no complex exponentials in the real world. Instead, we'd like to know the output in response to a cosine input. Fortunately, a cosine is the sum of two complex exponentials:

$$x[n] = \cos(\omega n) = \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\omega n},$$

therefore,

$$y[n] = \frac{G(\omega)}{2}e^{j\omega n} + \frac{G(-\omega)}{2}e^{-j\omega n}$$

Response to a Cosine

What is $G(-\omega)$? Remember the definition:

$$G(\omega) = \sum_m g[m]e^{-j\omega m}$$

Replacing every ω with a $-\omega$ gives:

$$G(-\omega) = \sum_m g[m]e^{j\omega m}.$$

Notice that $g[m]$ is real-valued, so the only complex number on the RHS is $e^{j\omega m}$. But

$$e^{j\omega m} = \left(e^{-j\omega m}\right)^*$$

so

$$G(-\omega) = G^*(\omega)$$

Response to a Cosine

$$\begin{aligned}
y[n] &= \frac{G(\omega)}{2} e^{j\omega n} + \frac{G^*(\omega)}{2} e^{-j\omega n} \\
&= \frac{|G(\omega)|}{2} e^{j\angle G(\omega)} e^{j\omega n} + \frac{|G(\omega)|}{2} e^{-j\angle G(\omega)} e^{-j\omega n} \\
&= \frac{|G(\omega)|}{2} e^{j(\omega n + \angle G(\omega))} + \frac{|G(\omega)|}{2} e^{-j(\omega n + \angle G(\omega))} \\
&= |G(\omega)| \cos\left(\omega n + \angle G(\omega)\right)
\end{aligned}$$

Magnitude and Phase Responses

- The **Magnitude Response** $|G(\omega)|$ tells you by how much a pure tone at ω will be scaled.
- The **Phase Response** $\angle G(\omega)$ tells you by how much a pure tone at ω will be advanced in phase.

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Example: First Difference

Remember that the first difference, $y[n] = x[n] - x[n-1]$, is supposed to sort of approximate a derivative operator:

$$y(t) \approx \frac{d}{dt}x(t)$$

If the input is a cosine, what is the output?

$$\frac{d}{dt}\cos(\omega t) = -\omega \sin(\omega t) = \omega \cos\left(\omega t + \frac{\pi}{2}\right)$$

Does the first-difference operator behave the same way (multiply by a magnitude of $|G(\omega)| = \omega$, phase shift by $+\frac{\pi}{2}$ radians so that cosine turns into negative sine)?

Example: First Difference

Freqeuncy response of the first difference filter is

$$G(\omega) = 1 - e^{-j\omega}$$

Let's try to convert it to polar form, so we can find its magnitude and phase:

$$\begin{aligned} G(\omega) &= e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right) \\ &= e^{-j\frac{\omega}{2}}\left(2j\sin\left(\frac{\omega}{2}\right)\right) \\ &= \left(2\sin\left(\frac{\omega}{2}\right)\right)\left(e^{j\left(\frac{\pi-\omega}{2}\right)}\right) \end{aligned}$$

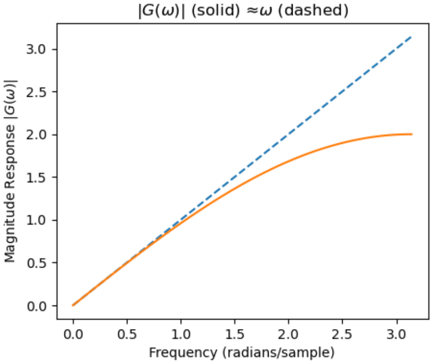
So the magnitude and phase response are:

$$|G(\omega)| = 2\sin\left(\frac{\omega}{2}\right)$$

$$\angle G(\omega) = \frac{\pi - \omega}{2}$$

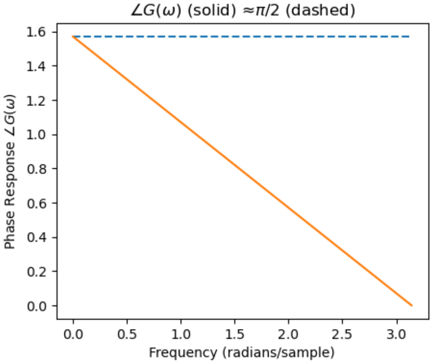
First Difference: Magnitude Response

Taking the derivative of a cosine scales it by ω. The first-difference filter scales it by $|G(\omega)| = 2\sin(\omega/2)$, which is almost the same, but not quite:



First Difference: Phase Response

Taking the derivative of a cosine shifts it, in phase, by $+\frac{\pi}{2}$ radians, so that the cosine turns into a negative sine. The first-difference filter shifts it by $\angle G(\omega) = \frac{\pi - \omega}{2}$, which is almost the same, but not quite.



First Difference: All Together

Putting it all together, if the input is $x[n] = \cos(\omega n)$, the output is

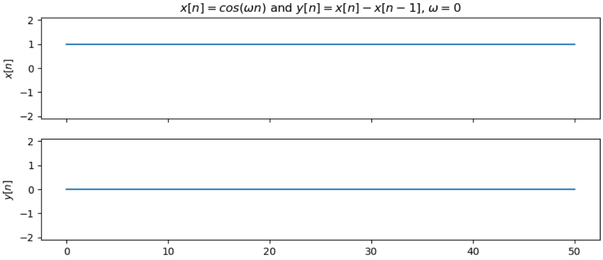
$$y[n] = |G(\omega)| \cos\left(\omega n + \angle G(\omega)\right) = 2\sin\left(\frac{\omega}{2}\right)\cos\left(\omega n + \frac{\pi - \omega}{2}\right)$$

- At frequency $\omega = 0$, the phase shift is exactly $\frac{\pi}{2}$, so the output is turned from cosine into -sine (but with an amplitude of 0!)
- At frequency $\omega = \pi$, the phase shift is 0! So the output is just a cosine at twice the amplitude.
- In between, $0 < \omega < \pi$,
  - The amplitude gradually increases, while
  - the phase gradually shifts, from a -sine function back to a cosine function.

First Difference: All Together

Putting it all together, if the input is $x[n] = \cos(\omega n)$, the output is

$$y[n] = |G(\omega)| \cos\left(\omega n + \angle G(\omega)\right) = 2\sin\left(\frac{\omega}{2}\right)\cos\left(\omega n + \frac{\pi - \omega}{2}\right)$$



Outline

Linearity

Filters are linear: if you scale the input, the output also scales. Thus if

$$x[n] = Ae^{j\omega_1 n} + Be^{j\omega_2 n},$$

then the output is the sum of the same two tones, each scaled by the corresponding frequency response:

$$y[n] = G(\omega_1)Ae^{j\omega_1 n} + G(\omega_2)Be^{j\omega_2 n}$$

Response to a Cosine

Linearity applies to complex numbers, not just real numbers! So if

$$x[n] = A\cos(\omega n + \theta) = \frac{A}{2}e^{j(\omega n+\theta)} + \frac{A}{2}e^{-j(\omega n+\theta)},$$

then

$$\begin{aligned} y[n] &= \frac{AG(\omega)}{2}e^{j(\omega n+\theta)} + \frac{AG^*(\omega)}{2}e^{-j(\omega n+\theta)} \\ &= \frac{A|G(\omega)|}{2}e^{j(\omega n+\theta+\angle G(\omega))} + \frac{A|G(\omega)|}{2}e^{-j(\omega n+\theta+\angle G(\omega))} \\ &= A|G(\omega)|\cos\left(\omega n + \theta + \angle G(\omega)\right) \end{aligned}$$

Outline

Summary

- **Tones in $\rightarrow$ Tones out**

$$x[n] = e^{j\omega n} \rightarrow y[n] = G(\omega)e^{j\omega n}$$
$$x[n] = \cos(\omega n) \rightarrow y[n] = |G(\omega)| \cos(\omega n + \angle G(\omega))$$
$$x[n] = A\cos(\omega n + \theta) \rightarrow y[n] = A|G(\omega)| \cos(\omega n + \theta + \angle G(\omega))$$

- where the **Frequency Response** is given by

$$G(\omega) = \sum_m g[m]e^{-j\omega m}$$