Lecture 9: Discrete-Time Fourier Transform

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ECE 401: Signal and Image Analysis, Fall 2020

Outline

1. Review: Frequency Response
2. Discrete Time Fourier Transform
3. Properties of the DTFT
4. Examples
5. Summary

What is Signal Processing, Really?

- When we process a signal, usually, we're trying to enhance the meaningful part, and reduce the noise.
- **Spectrum** helps us to understand which part is meaningful, and which part is noise.
- **Convolution** (a.k.a. filtering) is the tool we use to perform the enhancement.
- **Frequency Response** of a filter tells us exactly which frequencies it will enhance, and which it will reduce.

Review: Convolution

- A **convolution** is exactly the same thing as a **weighted local average**. We give it a special name, because we will use it very often. It's defined as:

$$y[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

- We use the symbol $*$ to mean "convolution:"

$$y[n] = g[n] * f[n] = \sum_m g[m]f[n-m] = \sum_m g[n-m]f[m]$$

Review: DFT & Fourier Series

Any periodic signal with a period of N samples, $x[n+N]=x[n]$, can be written as a weighted sum of pure tones,

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j2\pi kn/N},$$

which is a special case of the spectrum for periodic signals:

$$\omega_0=\frac{2\pi}{N}\frac{\text{radians}}{\text{sample}},\quad F_0=\frac{1}{T_0}\frac{\text{cycles}}{\text{second}},\quad T_0=\frac{N}{F_s}\frac{\text{seconds}}{\text{cycle}},\quad N=\frac{\text{samples}}{\text{cycle}},$$

and

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N}.$$

Tones in → Tones out

Suppose I have a periodic input signal,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N},$$

and I filter it,

$$y[n] = h[n] * x[n],$$

Then the output is a sum of pure tones, at the same frequencies as the input, but with different magnitudes and phases:

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{j2\pi kn/N}.$$

Frequency Response

Suppose we compute $y[n] = x[n] * h[n]$, where

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi kn/N}, \text{ and}$$

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{j2\pi kn/N}.$$

The relationship between $Y[k]$ and $X[k]$ is given by the frequency response:

$$Y[k] = H(k\omega_0)X[k]$$

where

$$H(\omega) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

Outline

Aperiodic

An "aperiodic signal" is a signal that is not periodic. Periodic acoustic signals usually have a perceptible pitch frequency; aperiodic signals sound like wind noise, or clicks.

- Music: strings, woodwinds, and brass are periodic, drums and rain sticks are aperiodic.
- Speech: vowels and nasals are periodic, plosives and fricatives are aperiodic.
- Images: stripes are periodic, clouds are aperiodic.
- Bioelectricity: heartbeat is periodic, muscle contractions are aperiodic.

Periodic

The spectrum of a periodic signal is given by its Fourier series, or equivalently in discrete time, by its discrete Fourier transform:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi kn}{N}}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}}$$

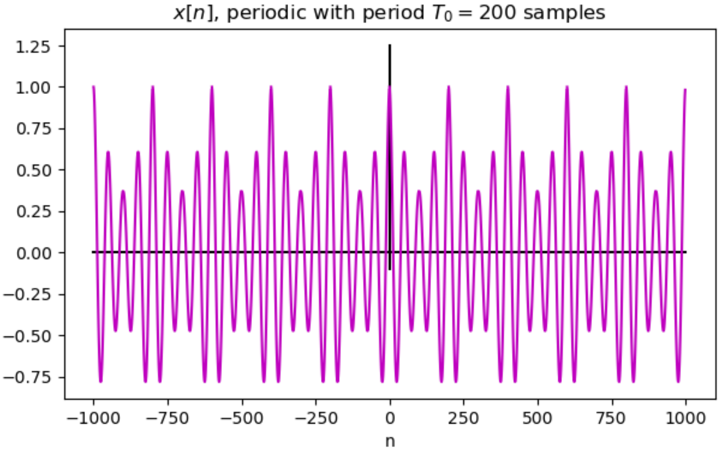
Aperiodic

The spectrum of an **aperiodic** signal we will now define to be exactly the same as that of a **periodic** signal except that, since it never repeats itself, its period has to be $N = \infty$:

$$x[n] \approx \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

$$X[k] \approx \lim_{N\to\infty} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}$$

An Aperiodic Signal is like a Periodic Signal with Period$= \infty$



Aperiodic

The spectrum of an **aperiodic** signal we will now define to be exactly the same as that of a **periodic** signal except that, since it never repeats itself, its period has to be $N=\infty$:

$$x[n]\approx\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi kn}{N}}$$

$$X[k]\approx\lim_{N\to\infty}\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi kn}{N}}$$

But what does that mean? For example, what is $\frac{2\pi k}{N}$? Let's try this definition: allow $k\to\infty$, and force ω to remain constant, where

$$\omega=\frac{2\pi k}{N}$$

Aperiodic

Let's start with this one:

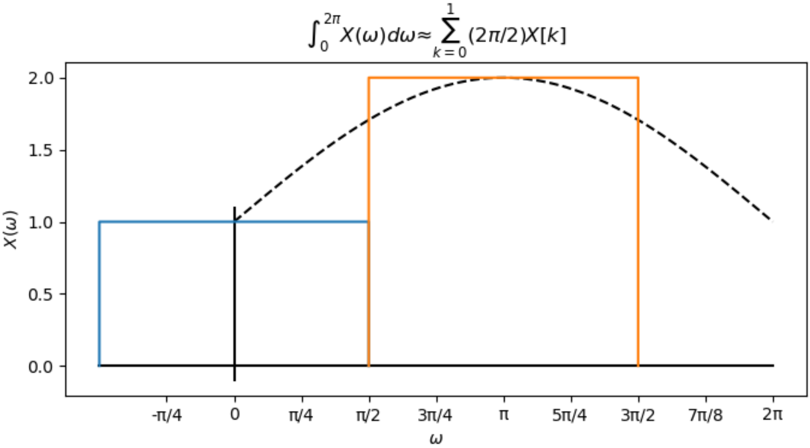
$$x[n] \approx \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}$$

Imagine this as adding up a bunch of tall, thin rectangles, each with a height of $X[k]$, and a width of $d\omega = \frac{2\pi}{N}$. In the limit, as $N \to \infty$, that becomes an integral:

$$\begin{aligned} x[n] &\approx \lim_{N\to\infty} \frac{1}{2\pi} \sum_{k=0}^{N-1} \frac{2\pi}{N} X[k] e^{j\frac{2\pi kn}{N}} \\ &= \frac{1}{2\pi} \int_{\omega=0}^{2\pi} X(\omega) e^{j\omega n} d\omega, \end{aligned}$$

where we've used $X(\omega) = X[k]$ just because, as $k \to \infty$, it makes more sense to talk about $X(\omega)$.

Approximating the Integral as a Sum



Periodic

Now, let's go back to periodic signals. Notice that $e^{j2\pi}=1$, and for that reason, $e^{j\frac{2\pi k(n+N)}{N}}=e^{j\frac{2\pi k(n-N)}{N}}=e^{j\frac{2\pi kn}{N}}$. So in the DFT, we get exactly the same result by summing over any complete period of the signal:

$$
\begin{aligned}
X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=1}^{N} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=-3}^{N-4} x[n]e^{-j\frac{2\pi kn}{N}} \\
&= \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} x[n]e^{-j\frac{2\pi kn}{N}}
\end{aligned}
$$

Aperiodic

Let's use this version, because it has a well-defined limit as $N \to \infty$:

$$X[k] = \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} x[n] e^{-j\frac{2\pi kn}{N}}$$

The limit is:

$$\begin{aligned} X(\omega) &= \lim_{N\to\infty} \sum_{n=-\frac{(N-1)}{2}}^{\frac{N-1}{2}} x[n] e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \end{aligned}$$

Discrete Time Fourier Transform (DTFT)

So in the limit as $N \rightarrow \infty$,

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$X(\omega)$ is called the discrete time Fourier transform (DTFT) of the aperiodic signal $x[n]$.

Outline

Properties of the DTFT

In order to better understand the DTFT, let's discuss these properties:

0. Periodicity
1. Linearity
2. Time Shift
3. Frequency Shift
4. Filtering is Convolution

Property #4 is actually the reason why we invented the DTFT in the first place. Before we discuss it, though, let's talk about the others.

0. Periodicity

The DTFT is periodic with a period of 2π. That's just because $e^{j2\pi} = 1$:

$$X(\omega) = \sum_n x[n]e^{-j\omega n}$$

$$X(\omega + 2\pi) = \sum_n x[n]e^{-j(\omega+2\pi)n} = \sum_n x[n]e^{-j\omega n} = X(\omega)$$

$$X(\omega - 2\pi) = \sum_n x[n]e^{-j(\omega-2\pi)n} = \sum_n x[n]e^{-j\omega n} = X(\omega)$$

In fact, we've already used this fact. I defined the inverse DTFT in two different ways:

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}d\omega$$

Those two integrals are equal because $X(\omega + 2\pi) = X(\omega)$.

1. Linearity

The DTFT is linear:

$$z[n] = ax[n] + by[n] \quad \leftrightarrow \quad Z(\omega) = aX(\omega) + bY(\omega)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_n z[n]e^{-j\omega n} \\ &= a\sum_n x[n]e^{-j\omega n} + b\sum_n y[n]e^{-j\omega n} \\ &= aX(\omega) + bY(\omega) \end{aligned}$$

2. Time Shift Property

Shifting in time is the same as multiplying by a complex exponential in frequency:

$$z[n] = x[n - n_0] \quad \leftrightarrow \quad Z(\omega) = e^{-j\omega n_0} X(\omega)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_{n=-\infty}^{\infty} x[n - n_0] e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega (m+n_0)} \quad \text{(where } m = n - n_0\text{)} \\ &= e^{-j\omega n_0} X(\omega) \end{aligned}$$

3. Frequency Shift Property

Shifting in frequency is the same as multiplying by a complex exponential in time:

$$z[n] = x[n]e^{j\omega_0 n} \quad \leftrightarrow \quad Z(\omega) = X(\omega - \omega_0)$$

Proof:

$$\begin{aligned} Z(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{j\omega_0 n}e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\omega_0)n} \\ &= X(\omega - \omega_0) \end{aligned}$$

4. Convolution Property

Convolving in time is the same as multiplying in frequency:

$$y[n] = h[n] * x[n] \quad \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$

Proof: Remember that $y[n] = h[n] * x[n]$ means that $y[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$. Therefore,

$$\begin{aligned} Y(\omega) &= \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} h[m]x[n-m] \right) e^{-j\omega n} \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (h[m]x[n-m])\, e^{-j\omega m} e^{-j\omega(n-m)} \\ &= \left(\sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m} \right) \left(\sum_{(n-m)=-\infty}^{\infty} x[n-m] e^{-j\omega(n-m)} \right) \\ &= H(\omega)X(\omega) \end{aligned}$$

Outline

Impulse and Delayed Impulse

For our examples today, let's consider different combinations of these three signals:

$$f[n] = \delta[n]$$
$$g[n] = \delta[n-3]$$
$$h[n] = \delta[n-6]$$

Remember from last time what these mean:

$$f[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$g[n] = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \begin{cases} 1 & n = 6 \\ 0 & \text{otherwise} \end{cases}$$

DTFT of an Impulse

First, let's find the DTFT of an impulse:

$$f[n] = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} F(\omega) &= \sum_{n=-\infty}^{\infty} f[n] e^{-j\omega n} \\ &= 1 \times e^{-j\omega 0} \\ &= 1 \end{aligned}$$

So we get that $f[n] = \delta[n] \leftrightarrow F(\omega) = 1$. That seems like it might be important.

DTFT of a Delayed Impulse

Second, let's find the DTFT of a delayed impulse:

$$g[n] = \begin{cases} 1 & n = 3 \\ 0 & \text{otherwise} \end{cases}$$

$$G(\omega) = \sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}$$

$$= 1 \times e^{-j\omega 3}$$

So we get that

$$g[n] = \delta[n-3] \leftrightarrow G(\omega) = e^{-j3\omega}$$

Similarly, we could show that

$$h[n] = \delta[n-6] \leftrightarrow H(\omega) = e^{-j6\omega}$$

Time Shift Property

Notice that

$$g[n] = f[n-3]$$
$$h[n] = g[n-3].$$

From the time-shift property of the DTFT, we can get that

$$G(\omega) = e^{-j3\omega}F(\omega)$$
$$H(\omega) = e^{-j3\omega}G(\omega).$$

Plugging in $F(\omega) = 1$, we get

$$G(\omega) = e^{-j3\omega}$$
$$H(\omega) = e^{-j6\omega}.$$

Convolution Property and the Impulse

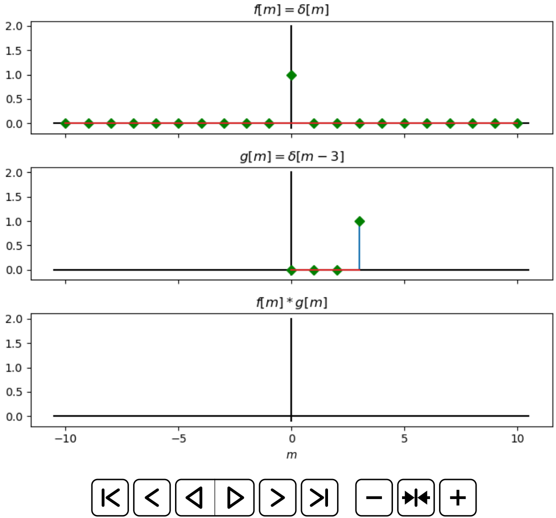
Notice that, if $F(\omega) = 1$, then anything times $F(\omega)$ gives itself again. In particular,

$$G(\omega) = G(\omega)F(\omega)$$
$$H(\omega) = H(\omega)F(\omega)$$

Since multiplication in frequency is the same as convolution in time, that must mean that

$$g[n] = g[n] * \delta[n]$$
$$h[n] = h[n] * \delta[n]$$

Convolution Property and the Impulse



Convolution Property and the Delayed Impulse

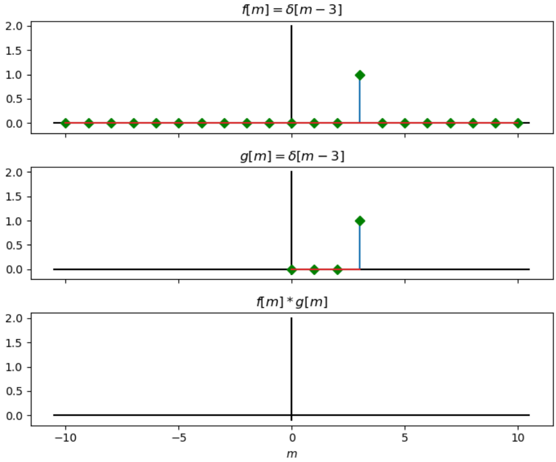
Here's another interesting thing. Notice that $G(\omega) = e^{-j3\omega}$, but $H(\omega) = e^{-j6\omega}$. So

$$\begin{aligned} H(\omega) &= e^{-j3\omega} e^{-j3\omega} \\ &= G(\omega) G(\omega) \end{aligned}$$

Does that mean that:

$$\delta[n-6] = \delta[n-3] * \delta[n-3]$$

Convolution Property and the Delayed Impulse



Outline

Summary

The DTFT (discrete time Fourier transform) of any signal is $X(\omega)$, given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Particular useful examples include:

$$f[n] = \delta[n] \leftrightarrow F(\omega) = 1$$
$$g[n] = \delta[n-n_0] \leftrightarrow G(\omega) = e^{-j\omega n_0}$$

Properties of the DTFT

Properties worth knowing include:

0. Periodicity: $X(\omega + 2\pi) = X(\omega)$
1. Linearity:

$$z[n] = ax[n] + by[n] \leftrightarrow Z(\omega) = aX(\omega) + bY(\omega)$$

2. Time Shift: $x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$
3. Frequency Shift: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
4. Filtering is Convolution:

$$y[n] = h[n] * x[n] \leftrightarrow Y(\omega) = H(\omega)X(\omega)$$