Lecture 20: Wiener Filter

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ECE 401: Signal and Image Analysis, Fall 2020

Outline

Three Types of Averages

We've been using three different types of averaging:

- **Expectation = Averaging across multiple runs of the same experiment**. If you run the random number generator many times, to generate many different signals $x[n]$, and then you compute the autocorrelation $r_{xx}[n]$ for each of them, then the average, across all of the experiments, converges to $E[r_{xx}[n]]$.
- **Averaging across time**.
- **Averaging across frequency**.

Three Types of Averages

Parseval's theorem says the total energy across time is the same as the average energy across frequency. That's true for either **actual energy** or **expected energy**:

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$E\left[\sum_{n=-\infty}^{\infty} x^2[n]\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} E\left[|X(\omega)|^2\right] d\omega$$

Things to know about expectation

There are only three things you need to know about expectation:

1. Definition: Expectation is the average across multiple runs of the same experiment.
2. Linearity: Expectation is linear.
3. Correlation: The expected product of two random variables is their **correlation**. If the expected product is the product of the expected values, the variables are said to be **uncorrelated**.

Expectation is Linear

The main thing to know about expectation is that it's linear. If x and y are random variables, and a and b are deterministic (not random), then

$$E\left[ax + by\right] = aE\left[x\right] + bE\left[y\right]$$

Correlated vs. Uncorrelated Signals

Uncorrelated random variables are variables x and y such that

$$\textbf{Uncorrelated RVs:} \quad E[xy] = E[x]E[y]$$

That doesn't work for correlated random variables:

$$\textbf{Correlated RVs:} \quad E[xy] \neq E[x]E[y]$$

Outline

1. Averaging and Expectation
2. Review: Noise
3. Wiener Filter
4. Summary

Wiener's Theorem and Parseval's Theorem

- Wiener's theorem says that the power spectrum is the DTFT of autocorrelation:

$$r_{xx}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xx}(\omega)e^{j\omega n}d\omega$$

- Parseval's theorem says that average power in the time domain is the same as average power in the frequency domain:

$$r_{xx}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xx}(\omega)d\omega$$

Filtered Noise

If $y[n] = h[n] * x[n]$, $x[n]$ is any noise signal, then

$$r_{yy}[n] = r_{xx}[n] * h[n] * h[-n]$$
$$R_{yy}(\omega) = R_{xx}(\omega)|H(\omega)|^2$$

White Noise and Colored Noise

If $x[n]$ is zero-mean unit variance white noise, and $y[n] = h[n] * x[n]$, then

$$E\left[r_{xx}[n]\right] = \delta[n]$$
$$E\left[R_{xx}(\omega)\right] = 1$$
$$E\left[r_{yy}[n]\right] = h[n] * h[-n]$$
$$E\left[R_{yy}(\omega)\right] = |H(\omega)|^2$$

Outline

Signals in Noise

Suppose you have

$$x[n] = s[n] + v[n]$$

- $s[n]$ is the signal — the part you want to keep.
- $v[n]$ is the noise — the part you want to get rid of. We call it $v[n]$ because $n[n]$ would be wierd, and because v looks kind of like the Greek letter ν, which sounds like n.

Task Statement

The goal is to design a filter $h[n]$ so that

$$y[n] = x[n] * h[n]$$

in order to make $y[n]$ as much like $s[n]$ as possible. In other words, let's minimize the mean-squared error:

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} (s[n] - y[n])^2$$

The Solution, if S and V are Known

If $s[n]$ and $v[n]$ are known, then we can solve the problem exactly. We want $Y(\omega) = S(\omega)$, where

$$Y(\omega) = H(\omega)X(\omega),$$

so we just need

$$H(\omega) = \frac{S(\omega)}{X(\omega)}$$

If S and V Not Known: This Solution Fails Badly!

If $s[n]$ and $v[n]$ are NOT known, can we make $Y(\omega) = E\left[S(\omega)|X(\omega)\right]$ by just solving

$$Y(\omega) = H(\omega)E\left[X(\omega)\right]?$$

Unfortunately, no, because $x[n] = s[n] + v[n]$ is a zero-mean random signal, so

$$E\left[X(\omega)\right] = 0$$

So dividing by $E\left[X(\omega)\right]$ is kind of a bad idea.

The Solution if S and V not known

OK, if S and V are unknown, here's a trick we can do to make the equation solvable:

$$S(\omega) = H(\omega)X(\omega)$$
$$S(\omega)X^*(\omega) = H(\omega)X(\omega)X^*(\omega)$$
$$E\left[S(\omega)X^*(\omega)\right] = H(\omega)E\left[X(\omega)X^*(\omega)\right]$$

which gives us

$$H(\omega) = \frac{E\left[S(\omega)X^*(\omega)\right]}{E\left[X(\omega)X^*(\omega)\right]}$$

Power Spectrum and Cross-Power Spectrum

Remember that the **power spectrum** is defined to be the Fourier transform of the **autocorrelation**:

$$R_{xx}(\omega) = \lim_{N\to\infty} \frac{1}{N}|X(\omega)|^2$$

$$r_{xx}[n] = \lim_{N\to\infty} \frac{1}{N}x[n] * x[-n]$$

In the same way, we can define the **cross-power spectrum** to be the Fourier transform of the **cross-correlation**:

$$R_{sx}(\omega) = \lim_{N\to\infty} \frac{1}{N}S(\omega)X^*(\omega)$$

$$r_{sx}[n] = \lim_{N\to\infty} \frac{1}{N}s[n] * x[-n]$$

The Wiener Filter

The **Wiener filter** is given by

$$H(\omega) = \frac{E\left[S(\omega)X^*(\omega)\right]}{E\left[|X(\omega)|^2\right]} = \frac{E\left[R_{sx}(\omega)\right]}{E\left[R_{xx}(\omega)\right]}$$

This creates a signal $y[n]$ that has the same statistical properties as the desired signal $s[n]$. Same expected energy, same expected correlation with $x[n]$, etc.

The Wiener Filter

$$Y(\omega) = \frac{E\left[R_{sx}(\omega)\right]}{E\left[R_{xx}(\omega)\right]} X(\omega) = \frac{E\left[S(\omega)X^*(\omega)\right]}{E\left[X(\omega)X^*(\omega)\right]} X(\omega)$$

- The numerator, $R_{sx}(\omega)$, makes sure that $y[n]$ is predicted from $x[n]$ as well as possible (same correlation, $E\left[r_{yx}[n]\right] = E\left[r_{sx}[n]\right]$).
- The denominator, $R_{xx}(\omega)$, divides out the noise power, so that $y[n]$ has the same expected power as $s[n]$.

Outline

Summary

Sorry no demos today! I'll try to have some on Thursday. Today we just had two key concepts: **Wiener filter** and **cross-power spectrum**:

$$H(\omega) = \frac{R_{sx}(\omega)}{R_{xx}(\omega)}$$

$$R_{sx}(\omega) = \lim_{N\to\infty} \frac{1}{N} S(\omega) X^*(\omega)$$

$$r_{sx}[n] = \lim_{N\to\infty} \frac{1}{N} s[n] * x[-n]$$