Lecture 21: Wiener Filter

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Outline

Wiener's Theorem and Parseval's Theorem

- Wiener's theorem says that the power spectrum is the DTFT of autocorrelation:

$$r_{xx}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{xx}(\omega)e^{j\omega n}d\omega$$

- Parseval's theorem says that energy in the time domain is the average of the energy spectrum:

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Filtered Noise

If $y[n] = h[n] * x[n]$, $x[n]$ is any signal, then

$$r_{yy}[n] = r_{xx}[n] * h[n] * h[-n]$$
$$R_{yy}(\omega) = R_{xx}(\omega)|H(\omega)|^2$$

The Wiener Filter

$$Y(\omega) = \frac{E\left[R_{sx}(\omega)\right]}{E\left[R_{xx}(\omega)\right]} X(\omega) = \frac{E\left[S(\omega)X^*(\omega)\right]}{E\left[X(\omega)X^*(\omega)\right]} X(\omega)$$

- The numerator, $R_{sx}(\omega)$, makes sure that $y[n]$ is predicted from $x[n]$ as well as possible (same correlation, $E\left[r_{yx}[n]\right] = E\left[r_{sx}[n]\right]$).
- The denominator, $R_{xx}(\omega)$, divides out the noise power, so that $y[n]$ has the same expected power as $s[n]$.

Power Spectrum and Cross-Power Spectrum

Remember that the **power spectrum** is defined to be the Fourier transform of the **autocorrelation**:

$$R_{xx}(\omega) = \lim_{N\to\infty} \frac{1}{N}|X(\omega)|^2$$

$$r_{xx}[n] = \lim_{N\to\infty} \frac{1}{N} x[n] * x[-n]$$

In the same way, we can define the **cross-power spectrum** to be the Fourier transform of the **cross-correlation**:

$$R_{sx}(\omega) = \lim_{N\to\infty} \frac{1}{N} S(\omega)X^*(\omega)$$

$$r_{sx}[n] = \lim_{N\to\infty} \frac{1}{N} s[n] * x[-n]$$

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An Alternate Derivation of the Wiener Filter

The goal is to design a filter $h[n]$ so that

$$y[n] = x[n] * h[n]$$

in order to make $y[n]$ as much like $s[n]$ as possible. In other words, let's minimize the mean-squared error:

$$\mathcal{E} = \sum_{n=-\infty}^{\infty} E\left[(s[n] - y[n])^2\right]$$

Use Parseval's Theorem!

In order to turn the convolutions into multiplications, let's use Parseval's theorem!

$$
\begin{aligned}
\mathcal{E} &= \sum_{n=-\infty}^{\infty} E\left[(s[n]-y[n])^2\right]\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} E\left[|S(\omega)-Y(\omega)|^2\right] d\omega\\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} E\left[|S(\omega)-H(\omega)X(\omega)|^2\right] d\omega\\
\mathcal{E} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(E\left[S(\omega)S^*(\omega)\right] - H(\omega)E\left[X(\omega)S^*(\omega)\right]\right.\\
&\quad \left. - E\left[S(\omega)X^*(\omega)\right]H^*(\omega) + H(\omega)E\left[X(\omega)X^*(\omega)\right]H^*(\omega)\right) d\omega
\end{aligned}
$$

Now let's try to find the minimum, by setting

$$
\frac{d\mathcal{E}}{dH(\omega)} = 0
$$

Differentiate and Solve!

Differentiating by $H(\omega)$ (and pretending that $H^*(\omega)$ stays constant), we get

$$\frac{d\mathcal{E}}{dH(\omega)} = -E\left[X(\omega)S^*(\omega)\right]d\omega + E\left[X(\omega)X^*(\omega)\right]H^*(\omega)d\omega$$

So we can set $\frac{d\mathcal{E}}{dH(\omega)} = 0$ if we choose

$$H^*(\omega) = \frac{E\left[X(\omega)S^*(\omega)\right]}{E\left[|X(\omega)|^2\right]}$$

or, equivalently,

$$H(\omega) = \frac{E\left[S(\omega)X^*(\omega)\right]}{E\left[|X(\omega)|^2\right]} = \frac{E\left[R_{sx}(\omega)\right]}{E\left[R_{xx}(\omega)\right]}$$

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What is X made of?

So here's the Wiener filter:

$$H(\omega) = \frac{E\left[S(\omega)X^*(\omega)\right]}{E\left[|X(\omega)|^2\right]}$$

But now let's break it down a little. What's X? That's right, it's $S + V$ — signal plus noise.

$$\begin{aligned} H(\omega) &= \frac{E\left[S(\omega)(S^*(\omega) + V^*(\omega))\right]}{E\left[|X(\omega)|^2\right]} \\ &= \frac{E\left[|S(\omega)|^2\right] + E\left[S(\omega)V^*(\omega)\right]}{E\left[|X(\omega)|^2\right]} \\ &= \frac{E\left[R_{ss}(\omega)\right] + E\left[R_{sv}(\omega)\right]}{E\left[R_{xx}(\omega)\right]} \end{aligned}$$

What if S and V are uncorrelated?

In most real-world situations, the signal and noise are uncorrelated, so we can write

$$E\left[S(\omega)V^*(\omega)\right] = E\left[S(\omega)\right]E\left[V^*(\omega)\right] = 0$$

What if S and V are uncorrelated?

Similarly, if S and V are uncorrelated,

$$E\left[|X(\omega)|^2\right] = E\left[|S(\omega)+V(\omega)|^2\right]$$

$$= E\left[|S(\omega)|^2\right] + E\left[S(\omega)V^*(\omega)\right] + E\left[S^*(\omega)V(\omega)\right] + E\left[|V(\omega)|^2\right]$$

$$= E\left[|S(\omega)|^2\right] + E\left[|V(\omega)|^2\right]$$

Wiener Filter in the General Case

In the general case, the Wiener Filter is

$$H(\omega) = \frac{E\left[R_{sx}(\omega)\right]}{E\left[R_{xx}(\omega)\right]}$$

$$= \frac{E\left[R_{ss}(\omega)\right] + E\left[R_{sv}(\omega)\right]}{E\left[R_{ss}(\omega)\right] - E\left[R_{sv}(\omega)\right] - E\left[R_{vs}(\omega)\right] + E\left[R_{vv}(\omega)\right]}$$

Wiener Filter for Uncorrelated Noise

If noise and signal are uncorrelated,

$$H(\omega) = \frac{E\left[R_{ss}(\omega)\right]}{E\left[R_{xx}(\omega)\right]} = \frac{E\left[R_{ss}(\omega)\right]}{E\left[R_{ss}(\omega)\right] + E\left[R_{vv}(\omega)\right]}$$

Wiener Filter in the General Case

$$H(\omega) = \frac{E\left[R_{sx}(\omega)\right]}{E\left[R_{xx}(\omega)\right]}$$

- In the general case, the numerator captures the correlation between the **noisy signal**, $x[n]$, and the desired clean signal $s[n]$.
- The idea is to give $y[n]$ the same correlation. We can't make $y[n]$ equal $s[n]$ exactly, but we can give it the same statistical properties as $s[n]$: specifically, make it correlate with $x[n]$ the same way.

Wiener Filter for Correlated Noise

$$H(\omega) = \frac{E\left[R_{ss}(\omega)\right]}{E\left[R_{xx}(\omega)\right]}$$

- If $s[n]$ and $v[n]$ are uncorrelated, then the correlation between the clean and noisy signals is exactly equal to the autocorrelation of the clean signal:

$$E\left[r_{sx}[n]\right] = E\left[r_{ss}[n]\right]$$

- So in that case, the Wiener filter is just exactly the **desired, clean** power spectrum, $E\left[R_{ss}(\omega)\right]$, divided by the **given, noisy** power spectrum $E\left[R_{xx}(\omega)\right]$,

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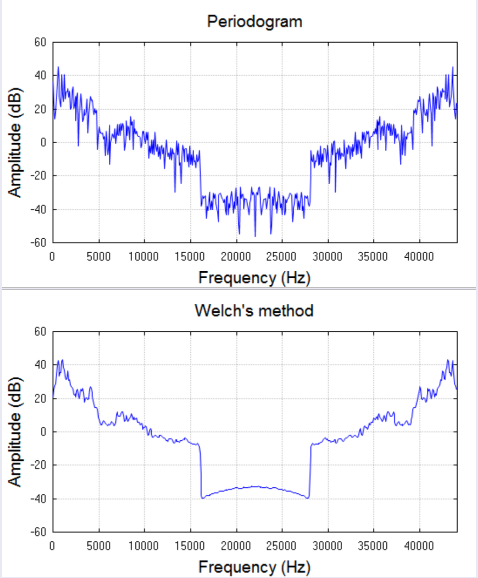
How can you compute expected value?

Finally: we need to somehow estimate the expected power spectra, $E[R_{ss}(\omega)]$ and $E[R_{xx}(\omega)]$. How can we do that?

- **Generative model:** if you know where the signal came from, you might have a pencil-and-paper model of its statistics, from which you can estimate $R_{ss}(\omega)$.
- **Multiple experiments:** If you have the luxury of running the experiment 1000 times, that's actually the best way to do it.
- **Welch's method:** chop the signal into a large number of small frames, computing $|X(\omega)|^2$ from each small frame, and then average. As long as the signal statistics don't change over time, this method works well.

Pros and Cons of Welch's Method

- **Con:** Because each $|X(\omega)|^2$ is being computed from a shorter window, you get less spectral resolution.
- **Pro:** Actually, less spectral resolution is usually a good thing. Micro-variations in the spectrum are probably noise, and should probably be smoothed away.



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Summary

- Wiener Filter in the General Case:

$$H(\omega) = \frac{E\left[R_{sx}(\omega)\right]}{E\left[R_{xx}(\omega)\right]}$$

- Wiener Filter for Uncorrelated Noise:

$$H(\omega) = \frac{E\left[R_{ss}(\omega)\right]}{E\left[R_{xx}(\omega)\right]}$$

- Welch's Method: chop the signal into frames, compute $|X(\omega)|^2$ for each frame, and then average them.