For exam 1, you need to know (1) phasors, (2) Fourier series (both discrete time and continuous time), (3) sampling, (4) upsampling. In other words, you should be able to do the following problems.

1.1 Phasors

Problem: $x(t) = \cos\left(2000\pi t + \frac{\pi}{6}\right) + \sin\left(2000\pi t - \frac{\pi}{3}\right)$. Use phasors to find the magnitude and phase of $x(t)$.

Solution:

$$x(t) = \Re\left\{e^{j2000\pi t}e^{j\pi/6} - je^{-j\pi/3}e^{j2000\pi t}\right\}$$

$$X = Y + Z, \quad Y = e^{j\pi/6} = \frac{\sqrt{3}}{2} + \frac{1}{2}j, \quad Z = -je^{-j\pi/3} = e^{--j5\pi/6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}j$$

$$Z = X + Y = 0, \quad |Z| = 0, \quad \angle Z = \text{undefined}$$

1.2 Fourier Series

Problem: $x(t) = |\cos(2\pi t)|$. Find T_0, Ω_0, and X_k.

Solution:

$$T_0 = \frac{1}{2}, \quad \Omega_0 = 4\pi$$

$$X_0 = \frac{1}{T_0}\int_{-1/4}^{1/4}|\cos(2\pi t)|dt = \frac{2}{\pi}$$

$$X_k = \frac{1}{T_0}\int_{-1/4}^{1/4}\left(\frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t}\right)e^{-jk4\pi t}dt$$

$$= \frac{1}{j2\pi(1-2k)}\left[e^{j2\pi(1-2k)(1/4)} - e^{j2\pi(1-2k)(-1/4)}\right] + \frac{1}{-j2\pi(1+2k)}\left[e^{-j2\pi(1+2k)(1/4)} - e^{-j2\pi(1+2k)(-1/4)}\right]$$

$$= \frac{1}{j2\pi(1-2k)}\left[j(-1)^k - (-j)(-1)^k\right] + \frac{1}{-j2\pi(1+2k)}\left[(-j)(-1)^k - j(-1)^k\right]$$

$$= \frac{(-1)^k}{\pi(1-2k)} + \frac{(-1)^k}{\pi(1+2k)} = \frac{2(-1)^k}{\pi(1-4k^2)}$$

1.3 Sampling

Problem: $x(t) = 3 + 2\cos(100\pi t) + \sin(250\pi t)$

(a) $x(t)$ is sampled at $F_{s,1} = 200$Hz, producing $x[n]$. Sketch the power spectrum of $x[n]$ over the range $0 \leq \omega < 2\pi$.

(b) $x[n]$ is passed through an ideal D/A at $F_{s,2} = 100$Hz, producing $y(t)$. Find $y(t)$.

Solution:

(a) $t = n/200$, so

$$x[n] = 3 + 2\cos\left(\frac{\pi n}{2}\right) + \sin\left(\frac{5\pi n}{4}\right)$$

$$= 3 + 2\cos\left(\frac{\pi n}{2}\right) + \sin\left(\frac{-3\pi n}{4}\right) = 3 + 2\cos\left(\frac{\pi n}{2}\right) - \sin\left(\frac{3\pi n}{4}\right)$$

$$= 3e^{j0} + e^{j\pi n/2} + e^{-j\pi n/2} + \frac{1}{2j}e^{-j3\pi n}4 - \frac{1}{2j}e^{j3\pi n}4$$

This has power $|X_\omega|^2 = 9$ at $\omega = 0$, $|X_\omega|^2 = 1$ at $\omega = \pi/2$, $|X_\omega|^2 = 1$ at $\omega = -\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}$, $|X_\omega|^2 = \frac{1}{4}$ at $\omega = \frac{3\pi}{4}$, and $|X_\omega|^2 = \frac{1}{4}$ at $\omega = -\frac{3\pi}{4} + 2\pi = \frac{5\pi}{4}$.

(b)

Ideal D/A only keeps the frequencies in $-\pi \leq \omega \leq \pi$, and replaces $t = nF_{s,2} = 100t$, so

$$y(t) = 3 + 2\cos(50\pi t) - \sin(75\pi t)$$

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1.4   Upsampling

$$x[m] = \sin(2\pi m/6), \quad \text{with period } M_0 = 6$$

$$y[n] = \begin{cases} x[m] & n = 3m \text{ for all integers } m \\ 0 & \text{otherwise} \end{cases}$$

Find the period N_0 of $y[n]$, and find its Fourier series coefficients Y_k for $0 \le k \le N_0 - 1$.

Solution: $x[m]$ has the Fourier series coefficients $X_1 = \frac{1}{2j}$, $X_5 = \frac{-1}{2j}$, and $X_0 = X_2 = X_3 = X_4 = 0$. $y[n]$ is periodic with $N_0 = 18$.

$$Y_k = \frac{1}{18} \sum_{n=0}^{17} y[n] e^{-j2\pi kn/18} = \frac{1}{3} \left(\frac{1}{6} \sum_{m=0}^{5} x[m] e^{-j2\pi km/6} \right) = \frac{1}{3} X_k$$

$$Y_k = \begin{cases} \frac{1}{6j} & k = 1, 7, 13 \\ \frac{-1}{6j} & k = 5, 11, 17 \\ 0 & \text{otherwise} \end{cases}$$