For exam 2, you need to know (1) impulse response, (2) linearity and time invariance, (3) frequency response, (4) discrete-time processing of continuous-time periodic signals, (5) DTFT. In other words, you should be able to do the following problems.

2.1 Impulse Response

Problem:

A particular system computes the average of 5 samples, minus the average of the previous five:

$$y[n] = \frac{1}{5}\sum_{m=0}^{4} x[n-m] - \frac{1}{5}\sum_{m=5}^{9} x[n-m]$$

Find the impulse response $h[n]$.

Solution:

Feed $\delta[n]$ into the system, and the output is:

$$h[n] = \begin{cases} \frac{1}{5} & 0 \le n \le 4 \\ -\frac{1}{5} & 5 \le n \le 9 \\ 0 & \text{otherwise} \end{cases}$$

2.2 Linearity and Time Invariance

Problem:

A particular system changes the sign of every second input sample, thus

$$y[n] = (-1)^n x[n]$$

Is this system linear? It is time-invariant? Prove your answers.

Problem:

$$\begin{aligned} x_1[n] &\rightarrow (-1)^n x_1[n] \\ x_2[n] &\rightarrow (-1)^n x_2[n] \\ ax_1[n] + bx_2[n] &\rightarrow (-1)^n (ax_1[n] + bx_2[n]) \\ &= ay_1[n] + by_2[n] \end{aligned}$$

So the system is linear.

$$\begin{aligned} x_1[n] &\rightarrow (-1)^n x_1[n] \\ y_1[n-m] &= (-1)^{n-m} x_1[n-m] \\ x_1[n-m] &\rightarrow (-1)^n x_1[n-m] \neq y_1[n-m] \end{aligned}$$

So the system is not time-invariant.

2.3 DT Processing of CT Signals

Problem:

A continuous-time periodic signal $x(t)$ with a period of $T_0 = 0.03$seconds is passed through an ideal anti-aliasing filter $H_a(\Omega)$, then sampled at $F_s = 1000$Hz to produce the discrete-time signal $x[n]$. The discrete-time signal $x[n]$ is then processed by the following system:

$$y[n] = \frac{1}{10}\sum_{m=0}^{9} x[n-m]$$

The signal $y[n]$ is then passed through an ideal D/A, at the same sampling frequency $F_s = 1000$Hz, to produce the signal $y(t)$. Let Y_k be the Fourier series coefficients of $y(t)$, and X_k those of $x(t)$.

1. Assume that $X_k \neq 0$ for all k; identify all k for which $Y_k = 0$.
2. Specify $|Y_k|$ in terms of $|X_k|$ for all non-zero Y_k.

Solution:

The harmonic frequencies of both $x(t)$ and $y(t)$ are $k\Omega_0 = 2\pi k/T_0 = 200\pi/3$ radians/second. The anti-aliasing filter eliminates all components $|\Omega| \geq \pi F_s = 1000\pi$, thus, all harmonics with $|k| \geq 15$. The other harmonics are mapped to $k\omega_0 = k\Omega_0/F_s = 2\pi k/30$. The discrete-time system has the frequency response

$$H(\omega) = e^{-j\omega(10-1)/2} \frac{\sin(\omega 10/2)}{10\sin(\omega/2)}$$

which has zeros at $\omega_\ell = 2\pi\ell/10$ for $\ell \neq 0$, thus every third harmonic of $x[n]$ is zeroed out. Thus

$$|Y_k| = \begin{cases} 0 & k = 3\ell, \quad \ell \neq 0 \\ 0 & |k| \geq 15 \\ \left|\frac{\sqrt{3}}{20\sin(k\pi/30)} X_k\right| & \text{otherwise} \end{cases}$$

2.4 DTFT

Problem:

A particular bandpass filter has the frequency response

$$F(\omega) = \begin{cases} 1 & \frac{\pi}{10} \leq |\omega| \leq \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

Find the impulse response $f[n]$.

Solution:

If your formula sheet has the form of an ideal lowpass filter on it, then the easiest way to solve this problem is to notice that

$$F(\omega) = G(\omega) - H(\omega)$$

where $G(\omega)$ is an ideal LPF with a cutoff of $\pi/4$, and $H(\omega)$ is an ideal LPF with a cutoff of $\pi/10$, thus

$$f[n] = \frac{sin(\pi n/4) - \sin(\pi n/10)}{\pi n}$$

Alternatively, you could use the inverse DTFT formula directly:

$$f[n] = \frac{1}{2\pi}\int_{-\pi/4}^{-\pi/10} e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\pi/10}^{\pi/4} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi j n}\left[e^{j\pi n/4} - e^{j\pi n/10} + e^{-j\pi n/10} - e^{j\pi n/4}\right] = \frac{\sin(\pi n/4) - \sin(\pi n/10)}{\pi n}$$