For exam 3, you need to know (1) Z transform of exponentials, relationship between Z transform, DTFT, and LCCDE, and the time-delay property of the Z transform (2) notch filters, (3) poles and zeros of the Z transform, stability, and the partial-fraction-expansion method of inverse Z transform.

2.1 Z transform

Problem:

Consider the following IIR filter:

$$y[n] = x[n] + \frac{1}{4}x[n-3] + \frac{1}{5}y[n-1] \tag{1}$$

1. Calculate the impulse response of this system, $h[n]$.
2. Calculate $H(z)$ by applying the Z transform formula directly to $h[n]$.
3. Calculate $H(z) = Y(z)/X(z)$ by applying the shift property of the Z transform to Eq. 1.
4. Use the relationship between Z-transform and DTFT to find the magnitude response, $|H(\omega)|$. Plot $|H(\omega)|$ as a function of ω, for $0 \le \omega \le \pi$. Label the amplitude at $\omega = 0$, $\omega = \pi/2$, and $\omega = \pi$. Notice that this one really isn't either a lowpass filter OR a highpass filter.

Solution:

1.

$$h[n] = \left(\frac{1}{5}\right)^n u[n] + \frac{1}{4}\left(\frac{1}{5}\right)^{n-3} u[n-3]$$

2.

$$H(z) = \frac{1}{1-(1/5)z^{-1}} - \frac{(1/4)z^{-3}}{1-(1/5)z^{-1}}$$

3.

$$H(z) = \frac{1+(1/4)z^{-3}}{1-(1/5)z^{-1}}$$

4.

$$|H(e^{j\omega})| = \left|\frac{1+(1/4)e^{-3j\omega}}{1-(1/5)e^{-j\omega}}\right|$$

$|H(e^{j0})| = 1.25/0.8$, $|H(e^{j\pi/2})| = |(1-(1/4)j)/(1-(1/5)j)| = \sqrt{1+(1/4)^2}/\sqrt{1+(1/5)^2} = \sqrt{425/416}$, $|H(e^{j\pi})| = 0.8/1.25$.

2.2 Notch Filter

Problem:

Suppose you have a signal $x[n] = s[n] + v[n]$ corrupted by a narrowband noise, $v[n]$, at the frequency $\pi/2$.

1. Find $H(z)$ for a notch filter, with a notch at $\pi/2$, and a bandwidth of $B = |\ln(0.99)|$ radians/sample.
2. Sketch the magnitude frequency response $|H(\omega)|$. Show the notch, show roughly the bandwidth of the notch, and show that it's $|H(\omega)| \approx 1$ at other frequencies.
3. Write the LCCDE that implements this filter.

Solution:

1.

$$H(z) = \frac{1+z^{-2}}{1+(0.99)^2 z^{-2}}$$

2. $|H(\omega)| \approx 1$ for all frequencies, except that at $|H(\pi/2)| = 0$, and $|H(0.5\pi + \ln(0.99))| = |H(0.5\pi - \ln(0.99))| \approx 1/\sqrt{2}$.

3.

$$y[n] = x[n] + x[n-2] - (0.99)^2 y[n-2]$$

2.3 Stability; Partial Fraction Expansion

Problem:

The vowel /i/, as in "feet," is produced by raising your tongue up toward your hard palate. The resulting vocal tract shape has resonant frequencies of F1=300, F2=1800, F3=2200, F4=3600Hz, with bandwidths of roughly B1=100, B2=150, B3=250, and B4=300Hz, respectively.

1. Assume a sampling rate of $F_s = 8000$Hz. Express the four resonant frequencies, and their bandwidths, in units of radians/sample.

2. This transfer function can be written as

$$H(z) = \frac{1}{\prod_{k=1}^{8}(1-p_k z^{-1})}$$

   Give the eight pole locations, p_1 through p_8. How do you know this filter is stable?

3. It is possible to implement this filter as

$$H(z) = H_1(z)H_2(z)H_3(z)H_4(z)$$

   where each of the filters $H_1(z)$ through $H_4(z)$ is at most second-order, and each one has real-valued coefficients. What is $H_1(z)$?

4. Write an LCCDE that implements $H_1(z)$.

5. Find the impulse response $h_1[n]$ of $H_1(z)$.

Solution:

1. $\omega_1 = \frac{3\pi}{40}$, $omega_2 = \frac{18\pi}{40}$, $\omega_3 = \frac{22\pi}{40}$, $\omega_4 = \frac{36\pi}{40}$, $\sigma_1 = \frac{2\pi}{80}$, $\sigma_2 = \frac{3\pi}{80}$, $\sigma_3 = \frac{5\pi}{80}$, $\sigma_4 = \frac{6\pi}{80}$.

2. $p_1 = e^{-\sigma_1 - j\omega_1}$, $p_2 = e^{-\sigma_1 + j\omega_1}$, $p_3 = e^{-\sigma_2 - j\omega_2}$, $p_4 = e^{-\sigma_2 + j\omega_2}$, $p_5 = e^{-\sigma_3 - j\omega_3}$, $p_6 = e^{-\sigma_3 + j\omega_3}$, $p_7 = e^{-\sigma_4 - j\omega_4}$, $p_8 = e^{-\sigma_4 + j\omega_4}$. The filter is stable because all of the poles are inside the unit circle, $|p_k| < 1$.

3.

$$H_1(z) = \frac{1}{(1-p_1 z^{-1})(1-p_2 z^{-1})} = \frac{1}{1 - e^{-\pi/40}\cos(3\pi/40)z^{-1} + e^{-2\pi/40}z^{-2}}$$

4.

$$y[n] = x[n] + e^{-\pi/40}\cos(3\pi/40)y[n-1] - e^{-2\pi/40}y[n-2]$$

5.

$$H_1(z) = \frac{1}{(1-p_2/p_1)(1-p_1/z)} + \frac{1}{(1-p_1/p_2)(1-p_2/z)}$$

   So

$$h_1[n] = \left(\frac{1}{1-p_2/p_1}\right)(p_2)^n u[n] \left(\frac{1}{1-p_1/p_2}\right)(p_1)^n u[n]$$
$$= \left(\frac{1}{1-e^{j6\pi/40}}\right) e^{j3n\pi/40} u[n] \left(\frac{1}{1-e^{-j6\pi/40}}\right) e^{-j3n\pi/40} u[n]$$