Lecture 10: Linearity and Time Invariance

ECE 401: Signal and Image Analysis

University of Illinois

2/28/2017

Outline

Convolution

Find $y[n] = x[n] * h[n]$ using graphical convolution, where

$$x[n] = \begin{cases} 1 & -1 \leq n \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h[n] = \delta[n] - \delta[n-1]$$

Outline

Linearity = Scaling and Adding

Suppose, when you put $x_k[n]$ into some system, $y_k[n]$ is the signal that comes out, for $1 \leq k \leq 3$. Then the system is **linear** if and only if

$$x_3[n] = ax_1[n] + bx_2[n] \Leftrightarrow y_3[n] = ay_1[n] + by_2[n]$$

Example

$$y[n] = x^2[n]$$

Then

$$y_3[n] = x_3^2[n] = a^2x_1^2[n] + 2abx_1[n]x_2[n] + b^2x_2^2[n]$$

but

$$ay_1[n] + by_2[n] = ax_1^2[n] + bx_2^2[n]$$

These are not equal, so the system is not linear.

Example

$$y[n] = nx[n]$$

Then

$$y_3[n] = nx_3[n] = anx_1[n] + bnx_2[n]$$

but

$$ay_1[n] + by_2[n] = anx_1[n] + bnx_2[n]$$

These are equal, so the system is linear.

Example

$$y[n] = \sum_{t=-\infty}^{n} x[t]$$

Then

$$y_3[n] = \sum_{t=-\infty}^{n} x_3[t] = a \sum_{t=-\infty}^{n} x_1[t] + b \sum_{t=-\infty}^{n} x_2[t]$$

but

$$ay_1[n] + by_2[n] = a \sum_{t=-\infty}^{n} x_1[t] + b \sum_{t=-\infty}^{n} x_2[t]$$

These are equal, so the system is linear.

Outline

Time Invariance = Shifting

Suppose, when you put $x_k[n]$ into some system, $y_k[n]$ is the signal that comes out, for $1 \le k \le 3$. Then the system is **time-invariant** if and only if

$$x_2[n] = x_1[n-m] \Leftrightarrow y_2[n] = y_1[n-m]$$

Example

$$y[n] = x^2[n]$$

Then

$$y_2[n] = x_2^2[n] = x_1^2[n-m]$$

but

$$y_1[n-m] = x_1^2[n-m]$$

These are equal, so the system is time-invariant.

Example

$$y[n] = nx[n]$$

Then

$$y_2[n] = nx_2[n] = nx_1[n-m]$$

but

$$y_1[n-m] = (n-m)x_1[n-m]$$

These are not equal, so the system is not time-invariant.

Example

$$y[n]=\sum_{t=-\infty}^{n} x[t]$$

Then

$$y_2[n]=\sum_{t=-\infty}^{n} x_2[t]=\sum_{t=-\infty}^{n} x_1[t-m]=\sum_{\tau=-\infty}^{n-m} x_1[\tau]$$

but

$$y_1[n-m]=\sum_{t=-\infty}^{n-m} x_1[t]$$

These are equal, so the system is time-invariant.

Outline

This system is linear and time invariant:

$$y[n] = \sum_{t=-\infty}^{n} x[t]$$

That means we can compute its output, in response to any input, using a convolve function (for example, using `np.convolve`):

$$y[n] = h[n] * x[n]$$

To find $h[n]$, we just put $x[n] = \delta[n]$ into the system, and see what comes out. What comes out is

$$h[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Remember, by the way, that this signal is called a unit step function, and is denoted $u[n]$, thus for this system, $h[n] = u[n]$.