Lecture 11: Frequency Response

ECE 401: Signal and Image Analysis

University of Illinois

3/2/2017

Outline

1. LTI Review
2. Frequency Response
3. Frequency Response of an Averager

LTI Review

Is this system linear? Is it time-invariant? Can you prove your answers?

$$y[n] = x[n] + x[n+5]$$

Outline

Frequency Response Example

Consider the system

$$y[n] = x[n-1] + x[n] + x[n+1]$$

Suppose the input is a cosine at some frequency ω, $x[n] = \cos(\omega n)$. Then the output is

$$y[n] = \cos(\omega(n-1)) + \cos(\omega n) + \cos(\omega(n+1))$$

Using the phasor method, we can write this as

$$y[n] = \Re\left\{e^{j\omega n}e^{-j\omega} + e^{j\omega n} + e^{j\omega n}e^{j\omega}\right\}$$
$$= \Re\left\{\left(e^{-j\omega} + 1 + e^{j\omega}\right)e^{j\omega n}\right\}$$
$$= \Re\left\{(1 + 2\cos(\omega))\,e^{j\omega n}\right\} = (1 + 2\cos\omega)\cos(\omega n)$$

So the output is a cosine at **exactly the same frequency**, but scaled by the frequency-dependent scaling factor

$$H(\omega) = 1 + 2\cos\omega$$

Frequency Response Derivation

Consider the LTI system

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

Suppose the input is $x[n] = e^{j\omega n}$. Then the output is

$$y[n] = \sum_{m=-\infty}^{\infty} h[m]e^{j\omega(n-m)} = e^{j\omega n} \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m}$$

$$= e^{j\omega n} H(\omega)$$

So the output is a complex exponential at **exactly the same frequency**, but scaled by the complex-valued, frequency-dependent constant

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m}$$

Frequency Response Definition

Frequency Response Definition

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m}$$

$x[n]$ =Complex Exponential

$$x[n] = e^{j\omega n} \rightarrow y[n] = H(\omega)x[n]$$

Frequency Response: Sinusoidal Inputs

$x[n]$ =Cosine

$$x[n] = \cos(\omega n) \rightarrow y[n] = |H(\omega)| \cos\left(\omega n + \angle H(\omega)\right)$$

$x[n]$ =Sine

$$x[n] = \sin(\omega n) \rightarrow y[n] = |H(\omega)| \sin\left(\omega n + \angle H(\omega)\right)$$

where $|H(\omega)|$ and $\angle H(\omega)$ are just the magnitude and phase of $H(\omega)$, i.e.,

$$H(\omega) = |H(\omega)|\, e^{j\angle H(\omega)}$$

Example: Averager = The Simplest Lowpass Filter

$$h[n] = \delta[n] + \delta[n-1]$$

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = 1 + e^{-j\omega}$$

$$= e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2}) = 2e^{-j\omega/2}\cos(\omega/2)$$

So

$$|H(\omega)| = 2\cos(\omega/2), \quad \angle H(\omega) = -\omega/2$$

Notice that $H(0) = 1$, while $H(\pi) = 0$, so this is a **lowpass filter**. Thus if $x[n] = \cos(\omega n)$ then

$$y[n] = \left(2\cos\left(\frac{\omega}{2}\right)\right)\cos\left(\omega\left(n - \frac{1}{2}\right)\right) = \begin{cases} 2\cos(\omega(n-1/2)) & \omega = 0 \\ 0 & \omega = \pi \end{cases}$$

Example: Euler Differencer = The Simplest Highpass Filter

$$h[n] = \delta[n] - \delta[n-1]$$

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = 1 - e^{-j\omega}$$

$$= e^{-j\omega/2}(e^{j\omega/2} - e^{-j\omega/2}) = 2je^{-j\omega/2}\sin(\omega/2)$$

So

$$|H(\omega)| = 2\sin(\omega/2), \quad \angle H(\omega) = \frac{\pi - \omega}{2}$$

Notice that $H(0) = 0$, while $H(\pi) = 1$, so this is a **highpass filter**.
Thus if $x[n] = \cos(\omega n)$ then

$$y[n] = \left(2\sin\left(\frac{\omega}{2}\right)\right)\cos\left(\omega\left(n - \frac{1}{2}\right) + \frac{\pi}{2}\right) = \begin{cases} 0 & \omega = \\ 2\sin(\omega(n-1/2)) & \omega = \end{cases}$$

Outline

Frequency Response of an Averager, in General

$$h[n] = u[n] - u[n-N] \quad \text{for some integer } N$$

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} = \sum_{m=0}^{N-1} e^{-j\omega m}$$

In order to solve this one, we need to use Zeno's paradox, which can be stated as follows. For any fraction a such that $|a| < 1$,

$$\sum_{m=0}^{\infty} a^m = \frac{1}{1-a}$$

(In the fable created by the ancient Greek philosopher Zeno of Elea, the fraction is $a = \frac{1}{2}$).

$$H(\omega) = \sum_{m=0}^{\infty} e^{-j\omega m} - \sum_{m=N}^{\infty} e^{-j\omega m}$$

$$= \sum_{m=0}^{\infty} e^{-j\omega m} - e^{-j\omega N} \sum_{m=0}^{\infty} e^{-j\omega m}$$

using Zeno's paradox, we convert this to

$$= \frac{1}{1 - e^{-j\omega}} - \frac{e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \left(\frac{e^{-j\omega N/2}}{e^{-j\omega/2}}\right)\left(\frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}}\right)$$

$$= e^{-j\omega\frac{(N-1)}{2}} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Frequency Response of an Averager

So the frequency response of this averager:

$$h[n] = u[n] - u[n-N] \quad \text{for some integer } N$$

is

$$H(\omega) = A(\omega)e^{-j\theta(\omega)}$$

where

$$A(\omega) = \left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right) \theta(\omega) = -\omega(N-1)/2$$

Frequency Response of an Averager

The **signed-amplitude response** of an averager has the folowing important characteristics

$$A(\omega) = \left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right)$$

We call that the **signed-amplitude response** because it can be either positive or negative; we only require that it should be real. So it's not exactly the same thing as the magnitude of the complex number.

$$A(\omega) = \begin{cases} N & \omega = 0 \\ 0 & \omega = 2\pi\ell/N, \quad \text{any integer } \ell \neq 0 \end{cases}$$

In particular, $H(\pi) = 0$, so this is a lowpass filter. We could say that the N-point averager is much more lowpass than the 2-point averager; its **cutoff frequency** is $\omega = 2\pi/N$.

Linear Phase

The **phase response** of an averager has the folowing important characteristic:

$$\theta(\omega) = -\omega(N-1)/2$$

Notice that this phase is a **linear** function of ω (we say the filter has **generalized linear phase.**). In general, a linear phase filter is one whose phase response looks like

$$\theta(\omega) = -\omega d$$

for any constant d. The constant d is called the **filter delay**, because

$$x[n] = \cos(\omega n) \rightarrow y[n] = A(\omega)\cos\left(\omega\left(n-d\right)\right)$$

So the filter acts as though it **delays** the input, by a delay of d samples.