Lecture 12: Sampled Systems

ECE 401: Signal and Image Analysis

University of Illinois

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Outline

Frequency Response Review

Find the frequency response of this system. Express it as $H(\omega) = A(\omega)e^{j\theta(\omega)}$ where $A(\omega)$ is something real-valued.

$$h[n] = \begin{cases} 1 & n = 0 \\ -1 & n = 2 \\ 0 & \text{otherwise} \end{cases}$$

Outline

Sampled Systems

Computers can be used to generate signal $y(t)$, given some input signal $x(t)$. The procedure is:

1. Filter $x(t)$ through an anti-aliasing filter, giving the filtered signal $\tilde{x}(t)$.
2. Sample $\tilde{x}(t)$ at a sampling rate of F_S, giving digital signal $x[n]$.
3. Process $x[n]$ to produce $y[n]$.
4. Pass $y[n]$ through a D/A to generate $y(t)$.

Outline

Anti-Aliasing Filter

- $x(t) = \cos(\Omega_1 t)$ gets sampled as $x[n] = \cos\left(\left(\frac{\Omega_1}{F_s}\right)n\right)$, which can equivalently be writen as $x[n] = \cos\left(\left(\frac{\Omega_1}{F_s} - 2\pi\ell\right)n\right)$ for any integer value of ℓ.
- If $x[n]$ is then passed back immediately through an ideal D/A, it will only equal $x(t)$ if $|\Omega_1| < \pi F_s$.
- Otherwise, it will be "aliased" to a new frequency, $\Omega_2 = \Omega_1 - 2\pi\ell F_s$, such that $|\Omega_2| < \pi F_s$.
- We can avoid aliasing by first removing, from $x(t)$, any stuff that would get aliased. This is done using a continuous-time lowpass filter:

$$x(t) \rightarrow \boxed{H_{LPF}(\Omega)} \rightarrow \tilde{x}(t)$$

where

$$H_{LPF}(\Omega) = \begin{cases} 1 & |\Omega| < \pi F_s \\ 0 & \text{otherwise} \end{cases}$$

Example: Fourier Series

Suppose $x(t)$ is periodic with period T_0, so that

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

Then after we filter it with the anti-aliasing filter, we get

$$\tilde{x}(t) = \sum_{k=-K}^{K} X_k e^{jk\Omega_0 t}$$

where K is the largest integer such that

$$K\Omega_0 < \pi F_s$$

Note $\tilde{x}(t) \neq x(t)$! The high-frequency harmonics (the harmonics that would get aliased by sampling) have been removed; they are gone forever. If you want to keep them, you need to use a higher sampling rate.

Outline

Sampling

We sample by measuring the signal F_s times/second:

$$x[n] = \tilde{x}\left(t = \frac{n}{F_s}\right)$$

. . . so if . . .

$$\tilde{x}(t) = \sum_{k=-K}^{K} X_k e^{jk\Omega_0 t}$$

then

$$x[n] = \sum_{k=-K}^{K} X_k e^{jk\omega_0 n}, \quad k\omega_0 = \frac{k\Omega_0}{F_s}$$

Outline

Filtering

Now we implement any processing we want, in discrete time. For example, if we have an LTI system:

$$y[n] = h[n] * x[n]$$

and

$$x[n] = \sum_{k=-K}^{K} X_k e^{jk\omega_0 n}$$

then

$$y[n] = \sum_{k=-K}^{K} H(k\omega_0) X_k e^{jk\omega_0 n}$$

where $H(k\omega_0)$ is the frequency response, $H(\omega)$, evaluated at the frequency of the k^{th} harmonic, which is $\omega = k\omega_0$.

Example: Averager

For example, suppose we average N consecutive samples:

$$y[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[n-m]$$

The frequency response is

$$H(\omega) = e^{-j\omega\left(\frac{N-1}{2}\right)}\frac{\sin(\omega N/2)}{N\sin(\omega/2)}$$

$$= \begin{cases} 1 & \omega = 0 \\ 0 & \omega = \frac{2\pi\ell}{N},\ \ 0 < \text{integer } \ell < N \\ \text{other values} & \text{other freqencies} \end{cases}$$

Example: Averager

So the output signal is

$$y[n] = \sum_{k=-K}^{K} Y_k e^{jk\omega_0 n}$$

where

$$Y_k = e^{-jk\omega_0\left(\frac{N-1}{2}\right)} \frac{\sin(k\omega_0 N/2)}{N\sin(k\omega_0/2)} X_k$$

For example

- $Y_0 = X_0$ exactly
- $Y_k = 0$ when $k\omega_0 = \frac{2\pi\ell}{N}$ for nonzero integers $\ell < N$.

Outline

Ideal D/A

Now we send the generated signal through an ideal D/A. If $x(t)$ was periodic, and if the digital processing was LTI, then the output will also be periodic with the same period:

$$y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{jk\Omega_0 t}$$

The relationship between X_k and Y_k is determined by two things: (1) the anti-aliasing filter and D/A, and (2) the frequency response.

Ideal D/A

- **Effect of the anti-aliasing filter and ideal D/A:** an ideal D/A can only generate signal up to frequencies of $\Omega = \pi F_s$, so it limits Y_k as follows:

$$Y_k = 0 \quad \text{for all} \quad |k\Omega_0| \geq \pi F_s$$

- **Effect of the digital filter:** For frequencies $|\Omega| < \pi F_s$, the following dimensional analysis works:

$$\left(\omega \frac{\text{radians}}{\text{sample}}\right) \times \left(F_s \frac{\text{samples}}{\text{second}}\right) = \left(\Omega \frac{\text{radians}}{\text{second}}\right)$$

Putting it all together,

$$Y_k = \begin{cases} H\left(\frac{k\Omega_0}{F_s}\right) X_k & |k\Omega_0| < \pi F_s \\ 0 & |k\Omega_0| \geq \pi F_s \end{cases}$$

Example: Averager

For example, suppose

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}$$

$$x(t) \rightarrow \boxed{H_{LPF}(\Omega)} \rightarrow \tilde{x}(t)$$

$$x[n] = \tilde{x}\left(t = \frac{n}{F_s}\right)$$

$$y[n] = \frac{1}{N}\sum_{m=0}^{N-1} x[n-m]$$

Example: Averager

Then $y(t)=\sum_{k=-\infty}^{\infty} Y_k e^{jk\Omega_0 t}$, where

- $Y_k = 0$ for $|k\Omega_0| > \pi F_s$
- For other k,

$$Y_k = e^{-j\frac{k\Omega_0}{F_s}\left(\frac{N-1}{2}\right)}\frac{\sin(k\Omega_0 N/2F_s)}{N\sin(k\Omega_0/2F_s)}X_k$$

In particular:

- $Y_0 = X_0$, they have the same DC offset
- $Y_k = 0$ for any k such that $k\Omega_0$ is a multiple of $2\pi F_s/N$. Another way to say the same thing: $Y_k = 0$ if kF_0 is a multiple of F_s/N. It's like the averager has laid down a list of zeros, which knock out any harmonics that happen to land at integer multiples of F_s/N.