Lecture 13: Discrete Time Fourier Transform (DTFT)

ECE 401: Signal and Image Analysis

University of Illinois

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Outline

Sampled Systems Review

The inputs and outputs are

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T_0}, \quad y(t) = \sum_{k=-\infty}^{\infty} Y_k e^{j2\pi kt/T_0}$$

Suppose that $T_0 = 0.001$s. Suppose that $x(t)$ is lowpass filtered by an ideal anti-aliasing filter with a cutoff of 5kHz, then sampled at $F_s = 10$kHz to create $x[n]$. $x[n]$ is then passed through a 5-sample averager to create $y[n]$:

$$y[n] = \frac{1}{5}\sum_{m=0}^{4} x[n-m]$$

Find Y_k in terms of X_k.

Outline

Frequency response and sine waves

$$x[n] = e^{jk\omega_0 n} \rightarrow y[n] = H(k\omega_0)e^{jk\omega_0 n}$$

Frequency response and periodic signals

$$x[n] = \sum_{k=-K}^{K} X_k e^{jk\omega_0 n} \rightarrow y[n] = \sum_{k=-K}^{K} Y_k e^{jk\omega_0 n}$$

$$Y_k = H(k\omega_0)X_k$$

What about non-periodic signals?

Can we extend that formula to non-periodic signals?

DTFT = "Frequency response" of $x[n]$

$$H(\omega) = \sum_{m=-\infty}^{\infty} h[m] e^{-j\omega m}, \quad \text{so define} \quad X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Convolution in Time = Multiplication in Frequency

$$y[n] = x[n] * h[n] \leftrightarrow Y(\omega) = X(\omega) H(\omega)$$

Proof: Convolution in Time equals Multiplication in Frequency

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

$$Y(\omega) = \sum_{n=-\infty}^{\infty} \left(\sum_{m=-\infty} h[m]x[n-m] \right) e^{-j\omega n}$$

$$= \sum_{(n-m)=-\infty}^{\infty} \left(\sum_{m=-\infty}^{\infty} h[m]x[n-m] \right) e^{-j\omega m} e^{-j\omega(n-m)}$$

$$= \left(\sum_{(n-m)=-\infty}^{\infty} x[n-m]e^{-j\omega(n-m)} \right) \left(\sum_{m=-\infty}^{\infty} h[m]e^{-j\omega m} \right)$$

Example: DTFT of a Triangle

Suppose we have $y[n] = h[n] * x[n]$, where

$$h[n] = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}, \quad x[n] = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & \text{otherwise} \end{cases}$$

Using graphical convolution, it's easy to show that

$$y[n] = \begin{cases} n & 0 \le n \le 5 \\ 10 - n & 5 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}$$

But what's $Y(\omega)$?

Example: DTFT of a Triangle

$$X(\omega) = H(\omega) = e^{-j\omega(5-1)/2}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

$$Y(\omega) = H(\omega)X(\omega) = e^{-j5\omega}\left(\frac{\sin(5\omega/2)}{\sin(\omega/2)}\right)^2$$

Example: $y[n] = x[n-3]$

Suppose $y[n] = x[n-3]$. This is the same as a system with

$$h[n] = \delta[n-3] \leftrightarrow H(\omega) = e^{-j\omega 3}$$

Therefore

$$Y(\omega) = e^{-j\omega 3} X(\omega)$$

Time Shift Property of the DTFT

$$y[n] = x[n-n_0] \leftrightarrow Y(\omega) = e^{-j\omega n_0} X(\omega)$$

Outline

Here's the most important new idea today. The DTFT has an inverse, just like the Fourier series.

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

- Continuous-Time Fourier Series (CTFS)
  - Time: Continuous (t), Periodic (T_0)
  - Frequency: Aperiodic, Discrete (k)
  - $$X_k = \frac{1}{T_0}\int x(t)e^{-jk\Omega_0 t}dt, \quad x(t) = \sum X_k e^{jk\Omega_0 t}$$
- Discrete-Time Fourier Series (DTFS)
  - Time: Discrete (n), Periodic (N_0)
  - Frequency: Periodic (N_0), Discrete (k)
  - $$X_k = \frac{1}{N_0}\sum x[n]e^{-jk\omega_0 n}, \quad x[n] = \sum X_k e^{jk\omega_0 n}$$
- Discrete-Time Fourier Transform (DTFT)
  - Time: Discrete (n), Aperiodic
  - Frequency: Periodic (2π), Continuous (ω)
  - $$X(\omega) = \sum x[n]e^{-j\omega n}, \quad x[n] = \frac{1}{2\pi}\int X(\omega)e^{j\omega n}d\omega$$

Outline

Ideal Lowpass Filter

$$H(\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Goal: can we implement this as $y[n] = h[h] * x[n]$ for some $h[n]$?

Ideal Lowpass Filter

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\left(\frac{1}{jn}\right)\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

$$= \frac{2j\sin(\omega_c n)}{2j\pi n} = \frac{\sin(\omega_c n)}{\pi n}$$

The Magical Sinc Function

The sinc function (pronounced like "sink") is defined as:

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

It has the characteristics that

$$\text{sinc}(0) = \begin{cases} 1 & x = 0 \\ 0 & x = \ell\pi, \quad \text{any integer } \ell \text{ except } \ell = 0 \\ \text{other values} & \text{other values of } x \end{cases}$$

Rectangle in Time ↔ Sinc in Frequency

$$h[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \leftrightarrow H(\omega) = e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Sinc in Time ↔ Rectangle in Frequency

$$h[n] = \frac{\sin(\omega_c n)}{\pi n} \leftrightarrow H(\omega) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$